Preservation of interpolation by fibring

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1 Introduction

The method of fibring for combining logics as originally proposed by Gabbay [13, 14], includes some other methods as fusion [29] as a special case. Albeit fusion is the best developed mechanism, mainly in what concerns preservation of properties as soundness, weak completeness, semantic Craig interpolation and decidability (see [31, 17]), fibring in general raises some difficulties at the semantic level [26]. The general quest for *preservation* represents one of the main research trends in fibring.

Although preservation of soundness and completeness has been investigated in the context of propositional-based logics [32, 28], first-order quantification [27], higher-order features [9], non truth-functional semantics [4], sequent and other deduction calculi [16, 24], other forms of preservation are still to be investigated.

We outline here some results on preservation of interpolation in the context of propositional-based logics endowed with a Hilbert calculus coping with global and local derivability consequences (see [8] for further details).

What is now generally known as Craig interpolation is a heritage of the classical results by W. Craig [10] for first-order logic. Several abstractions have been considered either in proof-theoretical way (e.g. [5]) or in (non-constructive) model-theoretical style (e.g. for modal and positive logics as in [20, 21], for intuitionistic logic as in [15] and for hybrid logic as in [1, 2]). The importance of Craig interpolation for some fundamental problems of complexity theory as shown in [22] and further developed in [23] associates the rate of growth of the interpolant and measures of complexity.

Interpolation properties are known to be related with properties of model theory as exemplified by the correspondence between Craig interpolation and joint consistency properties for classical propositional logic. This correspondence is mediated in the classical case by finite algebraizability and by the familiar (global) metatheorem of deduction.

In the general case of deducibility relations, however, specially in those where the peculiarities of local and global deduction interfere, this correspondence opens difficult and challenging problems. We refer here to *careful reasoning* when the distinction of global and local deduction is relevant: careful reasoning may lead to other forms of interpolation even at propositional level. In the sequel we use the labels l-property and g-property when it makes sense to distinguish between local and global versions of that property; we write d-property when there is no distinction between them.

The importance of a general form of metatheorem of deduction is stressed, proving that Craig interpolation implies another form of interpolation proposed by S. Maehara [19], thus showing that the mediation of metatheorem of deduction plays a central role. Moreover, in certain specific cases, local interpolation implies global interpolation.

A result on preservation of Craig interpolation for fusion of modal logics restricted to global reasoning was obtained in [17]. Herein we study this question in a much broader sense for a wide-scoped fibring combinations covering global and local reasoning and encompassing several logics besides the modal ones. We establish sufficient conditions for preservation of interpolation by fibring using a translation of formulae from the fibring to the components. Along the way we show the preservation of a generalized version of the deduction metatheorem.

General techniques for obtaining interpolation are not known in general; Craig interpolation for example fails unexpectedly in all Łukasiewicz logics L_n with n finite or infinite see [18], and also in all Gödel logics G_n for $n \geq 4$, see [3]. Developing constructive proofs of interpolation is still a harder problem. We obtain here constructive methods of Craig interpolation for special logics as it is the case of some many-valued logics and logics of formal inconsistency (as studied in [6]).

2 Preliminaries

A *signature* C is an $\mathbb{N}$-indexed family of countable sets. The elements of each C_k are called *constructors* of arity k. Let $sL(C, \Xi)$ be the free algebra over C generated by Ξ. We denote by $\text{var}(\varphi)$ the set of elements of Ξ occurring in φ and $\text{var}(\Gamma) = \cup_{\gamma \in \Gamma} \text{var}(\gamma)$. A *substitution* is any map $\sigma : \Xi \to sL(C, \Xi)$. Substitutions can inductively extended to formulae: $\sigma(\gamma)$ is the formula where each $\xi \in \Xi$ is replaced by $\sigma(\xi)$ and also to sets: $\sigma(\Gamma) = \{\sigma(\gamma) : \gamma \in \Gamma\}$. When $\text{var}(\varphi) = \{\xi_1, \ldots, \xi_n\}$ and $\sigma(\xi_i) = \psi_i$ for $i = 1, \ldots, n$, we may use $\varphi(\psi_1, \ldots, \psi_n)$ to denote $\sigma(\varphi)$.

A *rule* over C is a pair $r = \langle \Theta, \eta \rangle$ where $\Theta \cup \{\eta\} \subseteq sL(C, \Xi)$ and Θ is finite. A *deductive system* is a triple $\mathcal{D} = \langle C, R_l, R_\text{g} \rangle$ where C is a signature and both R_l and R_g are sets of rules over C such that $R_l \subseteq R_\text{g}$. The rules in R_l are called *local rules* and those in R_g are called *global rules*. The distinction between local and global rules is imparted in the concept of careful reasoning and is reflected when investigating metatheoretical properties and their preservation.

A *global derivation* of $\varphi \in sL(C, \Xi)$ from $\Gamma \subseteq sL(C, \Xi)$, indicated by $\Gamma \vdash^{\text{g}}_{\mathcal{D}} \varphi$, is a sequence $\psi_1 \ldots \psi_n$ such that ψ_n is φ and each ψ_i is either an element of Γ or there are a rule $r = \langle \{\theta_1, \ldots, \theta_m\}, \eta \rangle \in R_\text{g}$ and a substitution σ such that ψ_i is $\sigma(\eta)$ and $\sigma(\theta_j)$ appears in $\psi_1 \ldots \psi_{i-1}$ for every $j = 1, \ldots, m$. A *local derivation* of $\varphi \in sL(C, \Xi)$ from $\Gamma \subseteq sL(C, \Xi)$, indicated by $\Gamma \vdash^{l}_{\mathcal{D}} \varphi$, is a sequence $\psi_1 \ldots \psi_n$ such that ψ_n is φ and each ψ_i is either an element of Γ, is globally derivable from the empty set or there are a rule $r = \langle \{\theta_1, \ldots, \theta_m\}, \eta \rangle \in R_l$ and a substitution σ such that ψ_i is $\sigma(\eta)$ and $\sigma(\theta_j)$ appears in $\psi_1 \ldots \psi_{i-1}$ for every $j = 1, \ldots, m$. We use the notation $\Gamma \vdash^{d}_{\mathcal{D}} \varphi$ when stating properties that hold either for global or for local derivation. We will extend the derivations to sets: $\Gamma \vdash^{\text{g}}_{\mathcal{D}} \Psi$ with $\Psi \subseteq sL(C, \Xi)$ iff $\Gamma \vdash^{\text{g}}_{\mathcal{D}} \psi$ for every $\psi \in \Psi$.

Several deduction metatheorems can be considered as indicated in [12]: they generalize the usual deduction metatheorems that require the existence of an implication in the signature. Herein, we consider extended versions taking into account global and local reasoning as follows: A deduction system $\mathcal{D}$ has the *d-metatheorem of deduction* (d-MTD) if there is a finite set of formulae $\Delta \subseteq sL(C, \{\xi_1, \xi_2\})$ such that if $\Gamma, \varphi_1 \vdash^{d}_{\mathcal{D}} \varphi_2$ then $\Gamma \vdash^{d}_{\mathcal{D}} \Delta(\varphi_1, \varphi_2)$. And it has the *d-metatheorem of modus ponens* (d-MTMP) if there is a finite set of formulae $\Delta \subseteq sL(C, \{\xi_1, \xi_2\})$ such that the converse holds.

3 Interpolation

We recast here some forms of interpolation taking into account the distinction between local and global deduction. A deductive system has the *d-extension interpolation property* (d-EIP) whenever $\Gamma, \Psi \vdash^d_{\mathcal{D}} \varphi$ implies that there is $\Gamma' \subseteq sL(C, \mathrm{var}(\Psi) \cup \mathrm{var}(\varphi))$ such that $\Gamma \vdash^d_{\mathcal{D}} \Gamma'$ and $\Gamma', \Psi \vdash^d_{\mathcal{D}} \varphi$, for every $\Gamma, \Psi \subseteq sL(C, \Xi)$ and $\varphi \in sL(C, \Xi)$. Although as mentioned before Łukasiewicz logics L_n with $n \geq 3$ do not have CIP they do enjoy EIP.

A deductive system has the *d-Craig interpolation property* (d-CIP) whenever $\Gamma \vdash^d_{\mathcal{D}} \varphi$ and $\mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi) \neq \emptyset$ implies that there is $\Gamma' \subseteq sL(C, \mathrm{var}(\Gamma) \cap \mathrm{var}(\varphi))$ such that $\Gamma \vdash^d_{\mathcal{D}} \Gamma'$ and $\Gamma' \vdash^d_{\mathcal{D}} \varphi$, for every $\Gamma \subseteq sL(C, \Xi)$ and $\varphi \in sL(C, \Xi)$.

The relevance of careful reasoning is measured by the fact that in some cases it is also possible to relate local and global CIP. That is the case of deductive systems which share with modal and first-order logics the important property that we call *careful-reasoning-by-cases*. A deduction system $\mathcal{D}$ is said to enjoy *careful-reasoning-by-cases* iff whenever $\Gamma \vdash^{\mathrm{g}}_{\mathcal{D}} \varphi$ then there is Ψ such that $\Gamma \vdash^{\mathrm{g}}_{\mathcal{D}} \Psi$, $\mathrm{var}(\Psi) \subseteq \mathrm{var}(\Gamma)$ and $\Psi \vdash^l_{\mathcal{D}} \varphi$.

Theorem 3.1 A deductive system $\mathcal{D}$ enjoying careful-reasoning-by-cases has g-Craig interpolation whenever it has l-Craig interpolation property.

A deductive system has the *d-Maehara interpolation property* (d-MIP) whenever $\Gamma, \Psi \vdash^d \varphi$ and $sL(C, \mathrm{var}(\Gamma) \cap (\mathrm{var}(\Psi) \cup \mathrm{var}(\varphi))) \neq \emptyset$ implies that there is $\Gamma' \subseteq sL(C, \mathrm{var}(\Gamma) \cap (\mathrm{var}(\Psi) \cup \mathrm{var}(\varphi)))$ such that $\Gamma \vdash^d \Gamma'$ and $\Gamma', \Psi \vdash^d \varphi$, for every $\Gamma, \Psi \subseteq sL(C, \Xi)$ and $\varphi \in sL(C, \Xi)$. It is proved in [11] that a deductive system has g-MIP iff it has g-EIP and g-CIP. Careful reasoning allows to the following improvement that together with the result in [11] shows that d-MTD and d-EIP are provable from each other.

Theorem 3.2 A deductive system with d-MTD, d-MTMP and d-CIP has d-MIP.

4 Preserving metaproperties

Given two deductive systems $\mathcal{D}'$ and $\mathcal{D}''$, their *fibring* is the deductive system $\mathcal{D}$ defined as follows: $C_k = C'_k \cup C''_k$ for every $k \in \mathbb{N}$, $R_l = {R_l}' \cup {R_l}''$ and $R_{\mathrm{g}} = {R_{\mathrm{g}}}' \cup {R_{\mathrm{g}}}''$. Observe that the deductive system induced by $\mathcal{D}$ is not the union (in the sense of [30]) of consequence systems induced by $\mathcal{D}'$ and $\mathcal{D}''$ neither for local nor for global derivation. Moreover taking $\Gamma' \subseteq sL(C', \Xi)$ and $\Gamma'' \subseteq sL(C'', \Xi)$ in general we get that ${\Gamma'}^{\vdash'^d} \subset {\Gamma'}^{\vdash^d}$ and ${\Gamma''}^{\vdash''^d} \subset {\Gamma''}^{\vdash^d}$.

In order to analyze the preservation of MTD it is easier to provide an alternative characterization involving derivations in the object logic. We start by presenting a necessary and sufficient condition for a deductive system to have MTMP.

Lemma 4.1 A deductive system has d-MTMP iff there is a finite set $\Delta \subseteq sL(C, \{\xi_1, \xi_2\})$ of formulae such that $\Delta, \xi_1 \vdash^d_{\mathcal{D}} \xi_2$.

Lemma 4.2 A deductive system with d-MTMP has d-MTD iff there is a finite set $\Delta \subseteq sL(C, \{\xi_1, \xi_2\})$ of formulae such that: (1) $\{\xi_1\} \vdash^l_{\mathcal{D}} \Delta(\xi_2, \xi_1)$; (2) $\vdash^d_{\mathcal{D}} \Delta(\xi_1, \xi_1)$; and (3) $\Delta(\xi_1, \theta_1) \cup \cdots \cup \Delta(\xi_1, \theta_m) \vdash^d_{\mathcal{D}} \Delta(\xi_1, \eta)$ for each rule $r = \langle \{\theta_1, \ldots, \theta_m\}, \eta \rangle \in R_l$.

Theorem 4.3 d-MTMP and d-MTD are preserved by fibring deductive systems with the same Δ.

The relationship between global and local Craig interpolation in the fibring can be investigated. The first step is to be able to *transform* derivations in the fibring $\mathcal{D}$ into derivations in the components $\mathcal{D}'$ and $\mathcal{D}''$ by using a "ghost" technique and transformation maps τ' and τ'' over the enrichments of $\mathcal{D}'$ and $\mathcal{D}''$ with the ghosts.

Theorem 4.4 Careful-reasoning-by-cases is preserved by fibring deductive systems.

A deductive system has *conjunction* if there is a constructor $\wedge \in C_2$ and $\langle\{(\xi_1 \wedge \xi_2)\}, \xi_1\rangle, \langle\{(\xi_1 \wedge \xi_2)\}, \xi_2\rangle, \langle\{\xi_1, \xi_2\}, (\xi_1 \wedge \xi_2)\rangle \in R_{\mathrm{g}}$.

Theorem 4.5 g-Craig interpolation is preserved by the fibring $\mathcal{D}$ of two deductive systems $\mathcal{D}'$ and $\mathcal{D}''$ with conjunction, g-MTMP, g-MTMP and an axiom that can be instanced with any finite number of variables.

For instance, in modal logic the axiom $(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1))$ can be instanced with any finite number of variables: the instance $((\wedge_{j=1}^{k} \xi_j) \Rightarrow (\xi_2 \Rightarrow (\wedge_{j=1}^{k} \xi_j)))$ includes $\xi_1, \ldots, \xi_k$.

Craig interpolation can also be investigated when no hypothesis are present and we are dealing with systems that have implication connective. A deductive system has the *implication connective* iff $\Rightarrow \in C_2$ and the d-MTD and the d-MTMP hold with $\Delta = \{(\xi_1 \Rightarrow \xi_2)\}$. A deductive system with implication has *imp-Craig interpolation* if $\vdash^{\mathrm{g}}_{\mathcal{D}} (\varphi_1 \Rightarrow \varphi_2)$ and $\mathrm{var}(\varphi_1) \cup \mathrm{var}(\varphi_2) \neq \emptyset$ then there is $\psi \in sL(C, \mathrm{var}(\varphi_1) \cup \mathrm{var}(\varphi_2))$ such that $\vdash^{\mathrm{g}}_{\mathcal{D}} (\varphi_1 \Rightarrow \psi)$ and $\vdash^{\mathrm{g}}_{\mathcal{D}} (\psi \Rightarrow \varphi_2)$.

Theorem 4.6 Imp-Craig interpolation is preserved by the fibring $\mathcal{D}$ of two deductive systems $\mathcal{D}'$ and $\mathcal{D}''$ both with the same implication and conjunction.

The preservation of l-Craig interpolation requires more assumptions on the component logics, namely a refinement of the notion of careful-reasoning-by-cases. A deductive system $\mathcal{D}$ has *localized-careful-reasoning-by-cases* if $\Gamma, \Psi \vdash^{\mathrm{g}}_{\mathcal{D}} \varphi$, Ψ is finite and with a derivation where rules in R_{g} are only applied to hypotheses in Ψ then there is a finite $\Omega \in sL(C, \mathrm{var}(\Psi))$ such that $\Omega \subseteq \Psi^{\vdash^{\mathrm{g}}_{\mathcal{D}}}$ and $\Omega \vdash^{l}_{\mathcal{D}} \varphi$. Both modal and first-order logics have this property.

Theorem 4.7 l-Craig interpolation is preserved by the fibring $\mathcal{D}$ of two deductive systems $\mathcal{D}'$ and $\mathcal{D}''$ both with localized-careful-reasoning-by-cases, conjunction, l-MTMP, l-MTD and an axiom that can be instanced with any finite number of variables.

Theorem 4.8 d-Maehara interpolation (d-MIP) is preserved by fibring deductive systems $\mathcal{D}'$ and $\mathcal{D}''$ both with d-MTMP, d-MTD, conjunction and an axiom that can be instanced with any number of variables (and with localized-careful-reasoning-by-cases).

5 Some logics with constructive Craig interpolation

Some constructive proofs of Craig interpolation can be given for deductive systems that enjoy certain properties. For this purpose we need a few semantic notions. A *matrix* is a triple $\langle B, \cdot, D\rangle$ where $m = \langle B, \cdot\rangle$ is an algebra over C (of values) and $D \subseteq B$ (the set of distinguished values). A *valuation* is any map from Ξ to B. The denotation of formula $[\![\varphi]\!]^m_v$ is defined inductively in the expected way. A formula φ is a *global semantic consequence* of Γ, written $\Gamma \vDash^{\mathrm{g}} \varphi$ if $[\![\varphi]\!]^m_v \in D$ whenever

$[\![\gamma]\!]_v^m \in D$ for every $\gamma \in \Gamma$. When a logic has a unique matrix up to isomorphism we may use $v(\varphi)$ to refer to $[\![\varphi]\!]_v^m$.

A deductive system has *binary supremum* with respect to global derivation iff there is $\vee \in C_2$ such that (i) $\xi_i \vdash^{\mathrm{g}}_{\mathcal{D}} (\xi_1 \vee \xi_2)$ and (ii) if $\xi_i \vdash^{\mathrm{g}}_{\mathcal{D}} \psi$ then $(\xi_1, \vee\xi_2) \vdash^{\mathrm{g}}_{\mathcal{D}} \psi$ for $i = 1, 2$. A deductive system with binary supremum has any finite suprema as well. We represent by $(\xi_1 \vee \ldots \xi_n)$ the suprema of $\xi_1 \ldots, \xi_n$.

A deductive system is said to be *suitable* iff and there are $\delta_i \in sL(C, \{\xi_1\})$ with $i = 1, \ldots, n$ satisfying the following conditions:

1. for every $\varphi \in sL(C, \Xi)$ and δ_i there is $\theta_i^\varphi \in sL(C, \Xi)$ such that: (a) $\mathrm{var}(\theta_i^\varphi) \subseteq \mathrm{var}(\varphi) \setminus \{\xi_1\}$; (b) $\varphi, \delta_i \vdash^{\mathrm{g}}_{\mathcal{D}} \theta_i^\varphi$ and $\delta_i, \theta_i^\varphi \vdash^{\mathrm{g}}_{\mathcal{D}} \varphi$; (c) $\varphi, \delta_i \vdash^{\mathrm{g}}_{\mathcal{D}} \psi$ for every $i = 1, \ldots, n$ then $\varphi \vdash^{\mathrm{g}}_{\mathcal{D}} \psi$.

2. If $\gamma_1, \gamma \vdash^{\mathrm{g}}_{\mathcal{D}} \gamma_2$ and $\mathrm{var}(\gamma) \cap \mathrm{var}(\gamma_1) = \mathrm{var}(\gamma) \cap \mathrm{var}(\gamma_2) = \emptyset$ then $\gamma_1 \vdash^{\mathrm{g}}_{\mathcal{D}} \gamma_2$.

Conditions 1(a) and (b) mean that we can separate the formula into two equivalent formulae without sharing variables. Condition 1(c) indicates that in order to analyze φ it is enough to analyze δ_i for every $i = 1, \ldots, n$. We call property 2 the *omitting symbols property* that holds in every free algebra $sL(C, \Upsilon)$ where Υ is a finite subset of the set of variables Ξ and $sL(C, \Xi)$ is the union of such algebras [30] (Section 1.1). In classical logic, the omitting symbols property holds iff γ is a contradiction. If the deductive system is complete with respect to matrix semantics we can provide sufficient conditions for 2. Observe that if $\varphi \vDash^{\mathrm{g}} \psi$, φ is satisfiable and $\mathrm{var}(\varphi) \cap \mathrm{var}(\psi) = \emptyset$ then $\vDash^{\mathrm{g}} \psi$.

Lemma 5.1 Let $\mathcal{D}$ be a complete deductive system having implication. Assume that $\mathrm{var}(\gamma) \cap \mathrm{var}(\gamma_1) = \mathrm{var}(\gamma) \cap \mathrm{var}(\gamma_2) = \emptyset$, γ is satisfiable and for every formula α, $v(\alpha) = v'(\alpha)$ whenever $v(\xi) = v'(\xi)$ for each $\xi \in \mathrm{var}(\alpha)$. If $\gamma_1, \gamma \vDash^{\mathrm{g}}_{\mathcal{D}} \gamma_2$ then $\gamma_1 \vDash^{\mathrm{g}}_{\mathcal{D}} \gamma_2$.

Theorem 5.2 Every suitable deductive system with binary supremum and implication has g-Craig interpolation.

Constructive Craig interpolation can also be given for deductive systems allowing the possibility of expressing all truth values at the syntactical level. A logic $\mathcal{L}$ is *syntactically faithful* if its deductive system has binary supremum $\vee$ and there are $\beta_1, \ldots, \beta_n$ depending at most upon the variables $\xi_1, \ldots, \xi_n$ such that $v(\beta_i) = b_i$ for every valuation v and for every truth value $b_i \in B$. Several finite-valued and non-truth functional logics share this property. Besides Rosser-Turquette deductive systems [25] and Post systems, several logics of formal inconsistency such as **mbC**, **bC**, **Ci** and da Costa's $\mathcal{C}_n$ for $n \in \mathbb{N}$ are also syntactically faithful, see [7], and thus enjoy g-Craig interpolation.

Theorem 5.3 Every syntactically faithful logic has g-Craig interpolation.

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