Fibring Modal First-Order Logics: Completeness Preservation*

Amílcar Sernadas[1] Cristina Sernadas[1] Alberto Zanardo[2]

[1] CLC/CMA, Departamento de Matemática, IST, Portugal
[2] Dipartimento di Matematica Pura ed Applicata, University of Padova, Italy

Abstract

Fibring is defined as a mechanism for combining logics with a first-order base, at both the semantic and deductive levels. A completeness theorem is established for a wide class of such logics, using a variation of the Henkin method that takes advantage of the presence of equality and inequality in the logic. As a corollary, completeness is shown to be preserved when fibring logics in that class. A modal first-order logic is obtained as a fibring where neither the Barcan formula nor its converse hold.

1 Introduction

Given the interest in the topic of combination of logics [2] and the significance of fibring [8, 9, 14] among the combination mechanisms, we have been following a research program directed at establishing preservation results on fibring. In [17] we established the preservation of completeness when fibring propositional based logics. Here, we address the same problem in the more challenging context of first-order based logics. At the same time, we attempt to assess to what extent the techniques of fibring can be used in the long standing issue of combination of modalities and quantification.

Extrapolating the definition of fibring to first-order based logics raises new technical problems at both the semantic and the deductive levels.

At the semantic level, the problem is to find a suitable abstraction of semantic structures encompassing a wide class of logics. Indeed, fibring appears as an operation on logics endowed with the same kind of semantics (the so called homogeneous scenario for the combination). So, we need a notion of semantics that encompasses as special cases logics as different as modal propositional logic and classical quantifier logic. To this end, we deal with quantifiers as special modalities for which assignments play the role of worlds. From the point of

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view of fibring, it is very natural to look at quantifiers as modalities. This attitude is fully developed here and we show that it has interesting and somewhat surprising consequences. Another key ingredient of our approach is the rigidity of variables while having non-constant domains. We want variables to be rigid designators because we are motivated by applications, such as the application of temporal quantifier logic to reactive system specification and knowledge representation, where it is essential to be able to compare values of flexible terms at different times using (rigid) variables. We want to be able to work within the more general setting of possibly non-constant domains, since this is a common feature of many recent approaches to modal quantifier logics. Finally, the techniques of fibring led us to work with very general notions of quantifiers and modalities, therefore encompassing a much wider class of logics. These features of our semantic approach make it rather different from other recent approaches to modal quantifier logic.

At the deductive level, the main new problem is the need to deal with side constraints in inference rules, like "term t is free for variable x in formula φ". The main idea is to deal with such provisos as meta-predicates on substitutions.

For illustration purposes, a modal first-order logic is presented in detail at both model and proof theoretic levels. Both the Barcan formula and its converse are shown to fail in this logic. Definability of properties related to these formulae is briefly investigated. Once fibring is defined, this logic is recovered as a fibring.

In order to establish conditions for the preservation of completeness by fibring first-order based logics, we first obtain a completeness theorem. The latter is proved using a variation of the Henkin method where we take advantage of having equality and inequality in the logic. The rigidity of variables also plays a crucial role here. The completeness theorem is proved to hold for a wide class of first-order based logics. Besides fullness and congruence, some reasonable assumptions are made on quantifiers and modalities, independently of each other. The preservation of completeness by fibring follows by showing that these assumptions are preserved by fibring.

In Section 2, the basic linguistic components of first-order based (fob) logics are introduced. Section 3 presents the semantic notions. Section 4 is dedicated to the semantics of an interesting modal first-order logic. Section 5 contains the notions on Hilbert calculi for fob logics. Section 6 is concentrated on the completeness theorem. Fibring of fob logics is defined in Section 7. In Section 8, the preservation of completeness by fibring is proved under some natural assumptions. Finally, in Section 9 an assessment is made of what has been achieved and what is still ahead.

2 First-order based signatures and languages

It is worthwhile to describe in detail the language of fob logics. That is, what we accept as being a fob signature and how the language is generated by a signature.

We assume given once and for all three denumerable sets: X (the set of (quantification) *variables*), Θ (the set of *term schema variables*) and Ξ (the set of *formula schema variables*). We also assume as fixed the *equality symbol* $=$ and the *inequality symbol* $\neq$. The schema variables (or meta variables) will be used for writing for example schematic inference rules, following the approach [14, 17].

In the envisaged notion of fob signature we should include, as expected, function and predicate symbols. We also include in the signature connectives, quantifiers and modalities since we want to encompass a wide variety of logics. Finally, for technical reasons explained in Section 6, we also include individual symbols as distinct from 0-ary function symbols (constants).

Definition 2.1 A *fob signature* is a tuple $\Sigma = \langle I, F, P, C, Q, O \rangle$ where:

- I is a set (of *individual symbols*);
- $F = \{F_k\}_{k \in \mathbb{N}}$ is a family of sets (of *function symbols*);
- $P = \{P_k\}_{k \in \mathbb{N}}$ is a family of sets (of *predicate symbols*);
- $C = \{C_k\}_{k \in \mathbb{N}}$ is a family of sets (of *connectives*);
- $Q = \{Q_k\}_{k \in \mathbb{N}^+}$ is a family of sets (of *quantifiers*);
- $O = \{O_k\}_{k \in \mathbb{N}^+}$ is a family of sets (of *modalities*). △

In order to avoid grammatical ambiguities, we assume that the sets P_0, C_0 and Ξ are pairwise disjoint, as well as the sets I, F_0, X and Θ. For the same reason, we also assume that, for each k in $\mathbb{N}^+$, the sets C_k and O_k are disjoint.

For the purpose of describing the sets of terms and formulae generated from a fob signature it is useful to make explicit the underlying two-sorted algebra. Let S denote the set $\{\tau, \phi\}$, where τ and ϕ are the (meta) sorts of terms and formulae, respectively. Given a fob signature Σ, we define the family $G = \{G_{\vec{s}\,s}\}_{\vec{s} \in S^*, s \in S}$ of sets of *generators* as follows:

- $G_{\epsilon\,\tau} = I \cup F_0 \cup X \cup \Theta \cup \{\theta^x_{\theta'} : \theta, \theta' \in \Theta,\ x \in X\}$;
- $G_{\tau^k\,\tau} = F_k$ for $k > 0$;
- $G_{\epsilon\,\phi} = P_0 \cup C_0 \cup \Xi \cup \{\xi^x_\theta : \xi \in \Xi,\ \theta \in \Theta,\ x \in X\}$;
- $G_{\tau^2\,\phi} = \{=, \neq\} \cup P_2$;
- $G_{\tau^k\,\phi} = P_k$ for $k \notin \{0, 2\}$;
- $G_{\phi^k\,\phi} = C_k \cup \{(qx) : q \in Q_k \ \&\ x \in X\} \cup O_k$ for $k > 0$;
- all other sets are empty.

Consider the S-sorted free algebra induced by G. We denote by $T(\Sigma, X, \Theta)$ the carrier of sort τ and refer to its elements as Σ-*terms* (or, simply, *terms*), and by $L(\Sigma, X, \Theta, \Xi)$ the carrier of sort ϕ and refer to its elements as Σ-*formulae* (or, simply, *formulae*). Furthermore, we denote by $T(\Sigma, X)$ and $L(\Sigma, X)$, respectively, the sets of terms and formulae written without schema variables.

Finally, we denote by $gT(\Sigma)$ the set of *ground terms*, i.e., terms without variables, and by $cL(\Sigma, X)$ the set of *closed formulae* which are defined in the usual way. For the sake of simplicity, we may denote $T(\Sigma, X)$, $gT(\Sigma)$, $L(\Sigma, X)$ and $cL(\Sigma, X)$ by T, gT, L and cL, respectively.

Finally, we conclude this section on language issues by introducing some notation concerning substitutions that act on the schema variables.

A Σ-*substitution* ρ maps each term schema variable θ to a term $\theta\rho$ in $T(\Sigma, X)$ and each formula schema variable ξ to a formula $\xi\rho$ in $L(\Sigma, X)$. We denote the set of all Σ-substitutions by $Sub(\Sigma)$.

A Σ-*schema substitution* σ maps each term schema variable θ to a schema term $\theta\sigma$ in $T(\Sigma, X, \Theta)$ and each formula schema variable ξ to a schema formula $\xi\sigma$ in $L(\Sigma, X, \Theta, \Xi)$. We denote the set of all Σ-schema substitutions by $sSub(\Sigma)$.

It should be evident how to extend a schema substitution or a substitution to the whole language. It is only worthwhile to explain what happens in the case of $\theta^x_{\theta'}$ and ξ^x_θ. For instance, $\xi^x_\theta\sigma$ is obtained from $\xi\sigma$ by substituting the term $\theta\sigma$ for x.

3 Semantics

The idea is to build all the semantic notions for fob logics from a suitably general notion of interpretation structure over a given fob signature. Such a structure should provide the means for interpreting all the symbols. Given the mixed nature of fob logics that include both quantifiers and modalities, one expects that such a structure should include "individuals", "assignments" and "worlds".

Moreover, looking at a fob structure as a kind of *fibring* (in the sense of [8]) of its quantifier and modal components, one is led to conceive it as a kind of two dimensional modal-like structure composed of "points", with one dimension running on the assignments (for quantifiers) and the other dimension running on the worlds (for proper modalities). Therefore, we want to know the value of an expression at each point. For each point u we need to know the corresponding assignment $\delta = \alpha(u)$ and world $w = \omega(u)$. The interpretation of some symbols will depend only on the assignments, while the interpretation of others will depend only on the worlds. For visualizing this approach look at Figure 1 below.

The semantics of quantification is established by looking at different points sharing the same world (by varying the assignment). Vice-versa, the semantics of modalities is obtained by looking at different points sharing the same assignment (by varying the world). In this way, quantifiers appear as modal operators with assignments playing the role of worlds.

This view makes it easy to provide a rigid semantics for variables. The value of a variable should depend only on the choice of the assignment. This implies that we must have a fixed universe of individuals across the different worlds. But, we may still vary the scope of quantification from one world to another, since we do not assume that the set of assignments at a given world is composed of all functions from variables to individuals.

Connectives can be expected to be independent of both assignments and worlds. However, we choose to be quite more general here for technical reasons (for proving the completeness theorem in Section 6).

Finally, function and predicate symbols are by default flexible (they may depend on the world at end). Of course, as usual they are constant (they do not depend on the assignment at hand). It is also convenient to have individual symbols that are both constant (independent of the assignment) and rigid (independent of the world).

Definition 3.1 A Σ-*structure* is a tuple $\langle U, \Delta, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [\,\cdot\,] \rangle$ with the following components:

- U is a nonempty set (of *points*);
- Δ is a nonempty set (of *assignments*) and W is a nonempty set (of *worlds*);
- $\alpha : U \to \Delta$ and $\omega : U \to W$;
- D is a nonempty set (of *individuals*);
- $\mathcal{E} \subseteq D^U$ is a set (of *individual concepts*) and $\mathcal{B} \subseteq 2^U$ is a set (of *truth values*), such that $U \in \mathcal{B}$;
- the *interpretation map* $[\,\cdot\,]$ is a function defined by means of the following clauses 1) to 9), where

$$\begin{aligned} U_\delta &= \{u \in U : \alpha(u) = \delta\}, & \mathcal{B}_\delta &= \{b \cap U_\delta : b \in \mathcal{B}\}, \\ U_w &= \{u \in U : \omega(u) = w\}, & \mathcal{B}_w &= \{b \cap U_w : b \in \mathcal{B}\}, \\ U_{w\delta} &= U_w \cap U_\delta, & \mathcal{B}_{w\delta} &= \{b \cap U_{w\delta} : b \in \mathcal{B}\}, \end{aligned}$$

1) $[x] = \{[x]_\delta\}_{\delta \in \Delta}$ where $[x]_\delta \in D$ for $x \in X$;
2) $[i] = \{[i]_\delta\}_{\delta \in \Delta}$ where $[i]_\delta \in D$ for $i \in I$, and $[i]_{\alpha(u)} = [i]_{\alpha(u')}$ whenever $u, u' \in U_w$ for some $w \in W$;
3) $[f] = \{[f]_w\}_{w \in W}$ where $[f]_w : D^k \to D$ for $f \in F_k$;
4) $[=] : D^2 \to 2$ is the diagonal relation;
5) $[\neq] : D^2 \to 2$ is the complement of the diagonal relation;
6) $[p] = \{[p]_w\}_{w \in W}$ where $[p]_w : D^k \to 2$ for $p \in P_k$;
7) $[c] = \{[c]_{w\delta}\}_{w \in W, \delta \in \Delta}$ where $[c]_{w\delta} : (\mathcal{B}_{w\delta})^k \to \mathcal{B}_{w\delta}$ for $c \in C_k$;
8) $[qx] = \{[qx]_w\}_{w \in W}$ where $[qx]_w : (\mathcal{B}_w)^k \to \mathcal{B}_w$ for $q \in Q_k$ and $x \in X$;
9) $[o] = \{[o]_\delta\}_{\delta \in \Delta}$ where $[o]_\delta : (\mathcal{B}_\delta)^k \to \mathcal{B}_\delta$ for $o \in O_k$.

Finally, the sets $\mathcal{E}$ and $\mathcal{B}$ considered above are assumed to be such that the following derived functions are well defined:

i) $\widehat{x} : \to \mathcal{E}$ by $\widehat{x}(u) = [x]_{\alpha(u)}$; $\widehat{i} : \to \mathcal{E}$ by $\widehat{i}(u) = [i]_{\alpha(u)}$;

ii) $\widehat{f} : \mathcal{E}^k \to \mathcal{E}$ by $\widehat{f}(e_1, \ldots, e_k)(u) = [f]_{\omega(u)}(e_1(u), \ldots, e_k(u))$;

iii) $\widehat{=} : \mathcal{E}^2 \to \mathcal{B}$ by $\widehat{=}(e_1, e_2)(u) = [=](e_1(u), e_2(u))$;

iv) $\widehat{\neq} : \mathcal{E}^2 \to \mathcal{B}$ by $\widehat{\neq}(e_1, e_2)(u) = [\neq](e_1(u), e_2(u))$;

v) $\widehat{p} : \mathcal{E}^k \to \mathcal{B}$ by $\widehat{p}(e_1, \ldots, e_k)(u) = [p]_{\omega(u)}(e_1(u), \ldots, e_k(u))$;

vi) $\widehat{c} : \mathcal{B}^k \to \mathcal{B}$ by
$\widehat{c}(b_1, \ldots, b_k)(u) = [c]_{\omega(u)\alpha(u)}(b_1 \cap U_{\omega(u)\alpha(u)}, \ldots, b_k \cap U_{\omega(u)\alpha(u)})(u)$;

vii) $\widehat{qx} : \mathcal{B}^k \to \mathcal{B}$ by
$\widehat{qx}(b_1, \ldots, b_k)(u) = [qx]_{\omega(u)}(b_1 \cap U_{\omega(u)}, \ldots, b_k \cap U_{\omega(u)})(u)$;

viii) $\widehat{o} : \mathcal{B}^k \to \mathcal{B}$ by
$\widehat{o}(b_1, \ldots, b_k)(u) = [o]_{\alpha(u)}(b_1 \cap U_{\alpha(u)}, \ldots, b_k \cap U_{\alpha(u)})(u)$. △

Each element $e \in \mathcal{E}$ is an individual concept (adapting from [6]): the denotation of the term "the president of country x" may vary with the point at hand – the assignment and the time (world) at hand. Similarly, each $b \in \mathcal{B}$ is a truth value: the denotation of the formula "the president of country x = y" may also vary with the point at hand. The standard choices for the sets $\mathcal{E}$ and $\mathcal{B}$ are D^U and $\wp U$, respectively. Having the possibility of other choices makes this structure "general" in the sense of [17], borrowing the notion of general frame in modal logic [1]. This added freedom is really necessary when in Section 6 we establish a completeness theorem which holds in a wide class of fob logics.

Like in many other approaches to modal quantifier logic, this semantics allows different domains of individuals at different worlds, notwithstanding the fact that in a structure there is a fixed global universe D of individuals. Local domains are derived concepts in our case. At each world w, we should consider the following two local domains:

- $D^{\mathcal{E}}_w = \{d \in D : \exists e{\in}\mathcal{E}\ \exists u{\in}U\ \omega(u) = w\ \&\ e(u) = d\}$;
- $D^{\Delta}_w = \{d \in D : \exists x{\in}X\ \exists u{\in}U\ \omega(u) = w\ \&\ [x]_{\alpha(u)} = d\}$.

The set $D^{\mathcal{E}}_w$ contains all possible values of terms at w. The set D^{Δ}_w contains all possible values of variables at w. Hence, D^{Δ}_w contains all individuals which are relevant when evaluating a quantification at w. Since variables are terms, we have that $D^{\Delta}_w \subseteq D^{\mathcal{E}}_w$. In the simplest cases, we have $D^{\Delta}_w = D^{\mathcal{E}}_w = D$, Δ is isomorphic to D^X and $\mathcal{E} = D^U$, as illustrated in Example 4.1. Observe that, in a logic with universal quantification, if $D^{\Delta}_w \neq D^{\mathcal{E}}_w$ then the formula $\forall x \psi \Rightarrow \psi^x_t$ can be falsified even if ψ does not contain any modality. But the formula $\forall x \psi \Rightarrow (E(t) \Rightarrow \psi^x_t)$ will be valid when the existence predicate E is interpreted at each world w as D^{Δ}_w (provided that no modalities are involved).

It is important to observe that, although the formula $\forall x \psi \Rightarrow (E(t) \Rightarrow \psi^x_t)$ has a free logic flavor, our semantics is rather different from those given for instance in [10, 12] for modal first-order logic. In particular, our approach to the issue of the Barcan formulae is different, as explained in detail in Example 4.2 below and further commented after Remark 4.3.

As anticipated before Definition 3.1, the interpretation $[x]$ depends only on the assignment at hand. The interpretation $[i]$ also depends only on the assignment, but, furthermore, it must be constant within a given world. Naturally, $[f]$ and $[p]$ depend only on the world at hand. Equality and inequality are given their standard interpretations.

On the other hand, one might expect $[c]$ to be invariant since that is the case in the most usual fob logic (modal first-order logic). However, we make it dependent on the pair world-assignment for technical reasons. This added freedom is again essential in Section 6 when proving the completeness theorem.

Concerning the interpretation of quantifiers, we made $[qx]$ dependent only on the world at hand, inspired by the functionality of the usual quantifiers, having in mind the possibility of different ranges of quantification on different worlds. Then, the interpretation $[o]$ of a modality o is easily understood as the dual. It depends only on the assignment at hand. This fibring style approach is novel and will be further commented upon in the next section where we study in detail the semantics of our first example of a fob logic.

It is worthwhile to extend these comments to the algebraic operations $\hat{\cdot}$ induced by the interpretation of the symbols. The definition of the functions $\widehat{f}$ and $\widehat{p}$ imply that the truth of formulae depends on the world at hand already at the atomic level (and not only as a consequence of the semantics for the modal operators). Indeed, functions and predicates are dealt with as *flexible* designators since their denotations may vary across worlds. On the other hand, the value $\widehat{x}(u)$ does not depend on $\omega(u)$, but only on the assignment $\alpha(u)$. This means that variables are assumed to be *rigid* designators since they preserve their values across worlds. The same applies to $\widehat{i}(u)$. Furthermore, the constraint $\widehat{i}(u) = \widehat{i}(u')$ whenever $u, u' \in U_w$ for some $w \in W$ imposes that individual symbols also do not change their values within a given world. For this reason we say that they are *constant* designators, besides being rigid. But note that individual symbols may still have different values in different points, as long as the points are in coordinatewise disjoint "clouds". For instance, in Figure 1, the set U is the union of the sets A, B, and C, and we have $\widehat{i}(u) = \widehat{i}(u')$ for $u, u' \in A \cup B$, and for $u, u' \in C$; however we might have $\widehat{i}(u) \neq \widehat{i}(u')$ for $u \in A \cup B$ and $u' \in C$. In this figure, the horizontal marked part of U represents the set of u such that $\alpha(u) = \delta$, and the vertical marked part of U represents the set of u such that $\omega(u) = w$.

Finally, observe that, in general, the set $U_{w\delta}$ is not a singleton set and hence U in general cannot be viewed as a subset of the Cartesian product $W \times \Delta$.

Given a Σ-structure it is straightforward to extend the interpretation to terms and formulae, and, from there, to define two kinds of satisfaction (global at the structure and local at a point).

Definition 3.2 Given a Σ-structure $s = \langle U, \Delta, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [\,\cdot\,]\rangle$:

- $[\![\cdot]\!]^s_\tau : T(\Sigma, X) \to \mathcal{E}$ is inductively defined by $[\![t]\!]^s_\tau = \widehat{t}$, for $t \in X \cup I$ and $[\![f(t_1, \ldots, t_k)]\!]^s_\tau = \widehat{f}([\![t_1]\!]^s_\tau, \ldots, [\![t_k]\!]^s_\tau)$, for $f \in F_k$, $k \geq 0$;

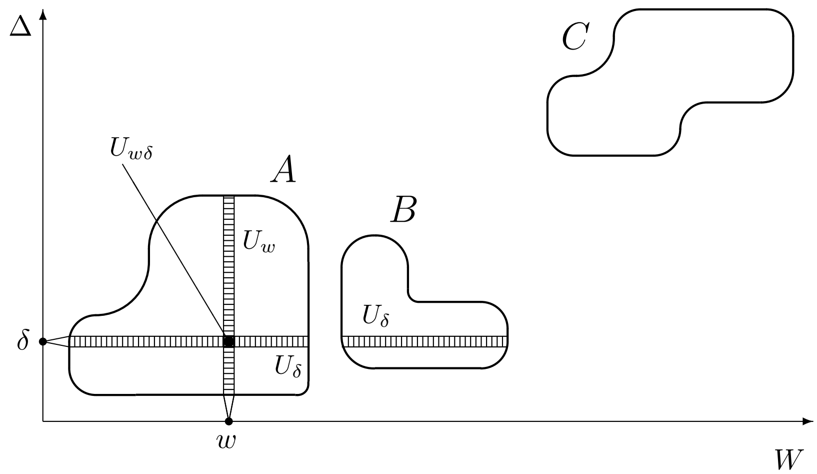

Figure 1: A fibred universe

- $[\![\cdot]\!]^s_\phi : L(\Sigma, X) \to \mathcal{B}$ is inductively defined in the same way as $[\![\cdot]\!]^s_\tau$, using the $\widehat{p}$'s, $\widehat{c}$'s, $\widehat{qx}$'s and $\widehat{o}$'s as well as taking into account $[\![\cdot]\!]^s_\tau$;
- $s \Vdash^\Sigma_{\mathrm{p}} \gamma$ iff $[\![\gamma]\!]^s_\phi = U$;
- for every $u \in U$, $su \Vdash^\Sigma_{\mathrm{d}} \gamma$ iff $u \in [\![\gamma]\!]^s_\phi$. △

Note that the *global satisfaction* and the *local satisfaction* are denoted by $\Vdash_{\mathrm{p}}$ and $\Vdash_{\mathrm{d}}$, respectively. This notation was chosen taking into account that global reasoning corresponds to proofs and local reasoning corresponds to derivations.

We might look directly at Σ-structures as models for the fob language over Σ. But we prefer to allow the possibility of working with other kinds of models as long as it is given a mechanism for extracting a Σ-structure from a model. The methodological advantage is obvious: we may then use the original models of an already known logic and just show how to get a structure from each of those models.

Definition 3.3 A Σ-*interpretation system* is a pair $\langle M, A\rangle$ where M is a class (of *models*) and A maps each $m \in M$ to a Σ-structure. △

Within the context of a Σ-interpretation system, we freely replace $A(m)$ by m, writing for instance $[\![\cdot]\!]^m_\tau$ instead of $[\![\cdot]\!]^{A(m)}_\tau$ and $mu \Vdash^\Sigma_{\mathrm{d}} \gamma$ for $A(m)u \Vdash^\Sigma_{\mathrm{d}} \gamma$.

Finally, we are ready to introduce the notion of the semantic component of a logic as follows.

Definition 3.4 An *interpretation framework* is a pair $\mathcal{S} = \langle Sig, S\rangle$ where:

- *Sig* is a class of fob signatures;
- S maps each $\Sigma \in Sig$ to a Σ-interpretation system.

The definition of the global and local entailments within a given interpretation framework brings no surprises.

Definition 3.5 Given an interpretation framework $\mathcal{S}$, we define for every $\Sigma \in Sig$, $\Gamma \subseteq L(\Sigma, X)$ and $\varphi \in L(\Sigma, X)$:

- $\Gamma \vDash^{\Sigma}_{\mathrm{p}} \varphi$ iff, for every $m \in M$ within $S(\Sigma)$, $m \Vdash^{\Sigma}_{\mathrm{p}} \varphi$ whenever $m \Vdash^{\Sigma}_{\mathrm{p}} \gamma$ for every $\gamma \in \Gamma$;
- $\Gamma \vDash^{\Sigma}_{\mathrm{d}} \varphi$ iff, for every $m \in M$ and $u \in U$ at $A(m)$ within $S(\Sigma)$, $mu \Vdash^{\Sigma}_{\mathrm{d}} \varphi$ whenever $mu \Vdash^{\Sigma}_{\mathrm{d}} \gamma$ for every $\gamma \in \Gamma$.

A detailed illustration of all these semantic concepts is provided in the next section where a novel semantics is proposed for the modal first-order logic.

4 Modal first-order logic

The following example serves two purposes. First, it is rich enough for illustrating the semantic concepts introduced in the previous section. Second, it shows that those concepts are general enough for encompassing a novel semantics of modal first-order logic where neither of the Barcan formulae is valid, notwithstanding the fact that the domain of individuals is constant.

Example 4.1 *Modal K first-order logic - interpretation framework.*

$$\mathcal{S}_{\mathrm{KFOL}} = \langle Sig, S \rangle$$

(i) The class *Sig* is composed of all fob signatures of the form

$$\Sigma(I, F, P) = \langle I, F, P, C, Q, O \rangle,$$

in which I is a set, $\langle F, P \rangle$ a fol (first-order logic) alphabet (function and relation symbols), and

- $C_1 = \{\neg\}, \quad C_2 = \{\wedge\}, \quad C_k = \emptyset$ for $k = 0$ or $k > 2$;
- $Q_1 = \{\forall\}, \quad O_1 = \{\Box\}, \quad Q_k = O_k = \emptyset$ for $k > 1$.

(ii) Each interpretation system $S(\Sigma(I, F, P)) = \langle M, A \rangle$ is defined as follows. We let M be the class of all tuples of the form

$$m = \langle D, W, R, V, \mathcal{I} \rangle$$

where:

- D and W are nonempty sets;
- $R = \{R_\delta\}_{\delta \in D^X}$ with each $R_\delta \subseteq W \times W$ (the *accessibility relation* at δ);
- $V(p) : W \to 2$ for $p \in P_0$;

- $\mathcal{I}(i) \in D$ for $i \in I$;
- $\mathcal{I}(f) = \{\mathcal{I}(f)_w\}_{w \in W}$ where $\mathcal{I}(f)_w : D^k \to D$ for $f \in F_k$;
- $\mathcal{I}(p) = \{\mathcal{I}(p)_w\}_{w \in W}$ where $\mathcal{I}(p)_w : D^k \to 2$ for $p \in P_k$ with $k > 0$.

Finally, for each model $m \in M$, we set:

$$A(m) = \langle U, \Delta, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [\,\cdot\,] \rangle$$

where:

- $U = W \times \Delta$ and $\Delta = D^X$;
- $\alpha(\langle w, \delta \rangle) = \delta$ and $\omega(\langle w, \delta \rangle) = w$;
- $\mathcal{E} = D^U$ and $\mathcal{B} = 2^U$;

(so that, the elements of $\mathcal{B}_w$ and $\mathcal{B}_\delta$ have respectively the form $(\{w\} \times \Delta')$ and $(W' \times \{\delta\})$ with $\Delta' \subseteq \Delta$ and $W' \subseteq W$)

- $[x]_\delta = \delta(x)$ and $[i]_\delta = \mathcal{I}(i)$;
- $[f] = \mathcal{I}(f)$ for $f \in F_k$, $[p] = \mathcal{I}(p)$ for $p \in P_k$ with $k > 0$, and $[p]_w = V(p)(w)$ for $p \in P_0$;
- $[\neg]_{w\delta}(b) = U_{w\delta} \setminus b$ and $[\wedge]_{w\delta}(b_1, b_2) = b_1 \cap b_2$;
- $[\forall x]_w(\{w\} \times \Delta')(\langle w, \delta \rangle) = 1$ iff $\delta' \in \Delta'$ for every $\delta' \in D^X$ such that δ' is x-equivalent to δ;
- $[\Box]_\delta(W' \times \{\delta\})(\langle w, \delta \rangle) = 1$ iff $w' \in W'$ for every $w' \in W$ such that $wR_\delta\, w'$.

Observe that, for each $w \in W$, the pair $\langle D, \mathcal{I}_w \rangle$ is a fol interpretation structure corresponding to a vertical fiber in Figure 1, and, for each $\delta \in D^X$, the triple $\langle W, R_\delta, V \rangle$ is a Kripke model corresponding to a horizontal fiber in the same figure.

According to Definition 3.1, for every $b \subseteq U$ and every $\langle w, \delta \rangle$ in U, we have

$$\widehat{\Box}(b)(\langle w, \delta \rangle) = [\Box]_\delta(b \cap U_\delta)(\langle w, \delta \rangle)$$

This means that $\langle w, \delta \rangle \in \widehat{\Box}(b)$, if and only if, for every $\langle w', \delta \rangle \in U_\delta$, if $wR_\delta\, w'$, then $\langle w', \delta \rangle \in b$. Setting

$$\langle w, \delta \rangle \widehat{R} \langle w', \delta' \rangle \quad \text{iff} \quad \delta = \delta' \text{ and } wR_\delta\, w',$$

we have that

$$\widehat{\Box}(b) \;=\; \{u : \forall u' \in U, u\widehat{R}u' \Rightarrow u' \in b\}$$

and hence $\langle \mathcal{B}, \widehat{\mathbf{f}}, \widehat{\mathbf{t}}, \widehat{\neg}, \widehat{\wedge}, \widehat{\vee}, \widehat{\Box} \rangle$ turns out to be a modal algebra, where $\widehat{\Box}$ is the modal operator induced by the relation $\widehat{R}$, $\widehat{\mathbf{f}} = \emptyset$ and $\widehat{\mathbf{t}} = U$.

A similar reasoning leads to view the interpretation of quantifiers $\forall x$ as the interpretation of a (vertical) S5 modality. For every $b \subseteq U$, the set $\widehat{\forall x}(b)$ is the set of all u such that, for every $u' \in U_{\omega(u)}$, if $\alpha(u)$ is x-equivalent to $\alpha(u')$, then

$u' \in b$. This means that $\langle \mathcal{B}, \widehat{\mathbf{f}}, \widehat{\mathbf{t}}, \widehat{\neg}, \widehat{\wedge}, \widehat{\vee}, \{\widehat{\forall x}\}_{x \in X}\rangle$ is a multi-modal algebra, where, for each variable x the operator $\widehat{\forall x}$ is induced by the relation $\widehat{R_x}$ defined by

$$\langle w, \delta\rangle \widehat{R_x} \langle w', \delta'\rangle \quad \text{iff} \quad w = w' \text{ and } \delta' \text{ is } x\text{-equivalent to } \delta$$

We can observe here that the semantics for $\Box$ has a horizontal dimension, namely, in order to evaluate a formula of the form $\Box\varphi$ at a given u, we consider the truth of φ at points u' such that $\alpha(u) = \alpha(u')$, which means that, in this evaluation, we always keep the same assignments. △

In the semantics for modal first-order logic we proposed above, the set of individuals is constant across worlds. However, the following example shows that making the accessibility relation dependent on the assignment is enough to invalidate both Barcan formulae.

Example 4.2 *Barcan formulae.* We show that none of the formulae $\forall x \Box\varphi \Rightarrow \Box\forall x\varphi$ and $\Box\forall x\varphi \Rightarrow \forall x\Box\varphi$ is valid in the semantics for modal first-order logic given above.

Let s_1 be the structure pictured in Figure 2 in which

$$W = \{w_1, w_2\} \quad \text{and} \quad D = \{d, d'\}\,.$$

According to Example 4.1, these definitions determine the sets U, Δ, $\mathcal{B}$, and $\mathcal{E}$, as well as the functions ω and α. Fix a variable x and, for all $u \in U$, set

$$R_{\alpha(u)} = \{\langle w_1, w_2\rangle\}\,,\ \text{if } \alpha(u)(x) = d\,; \qquad R_{\alpha(u)} = \emptyset\,,\ \text{otherwise}\,.$$

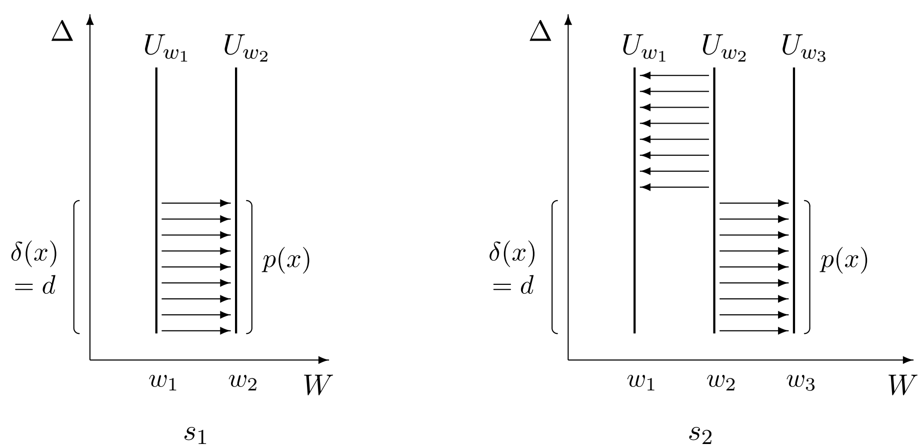

Figure 2: Barcan counterexamples

Assume that, in the signature at hand, P_1 contains an element p and let

$$[p]_{w_1}(d) = [p]_{w_1}(d') = 0, \quad [p]_{w_2}(d) = 1, \quad [p]_{w_2}(d') = 0$$

Thus, for every $u \in U$,

$$u \in [\![p(x)]\!]^{s_1}_{\phi} \quad \text{iff} \quad \omega(u) = w_2 \text{ and } \alpha(u)(x) = d \qquad (*)$$

This implies that, for every $u \in U_{w_1}$, $u \in [\![\Box p(x)]\!]^{s_1}_{\phi}$; in fact, either $\alpha(u)(x) \neq d$, and hence $R_{\alpha(u)}$ is empty, or $\alpha(u)(x) = d$, so that $R_{\alpha(u)} = \{\langle w_1, w_2\rangle\}$ and the claim is a consequence of $(*)$. Then, we can conclude $u \in [\![\forall x \Box p(x)]\!]^{s_1}_{\phi}$ for all $u \in U_{w_1}$.

Consider now any element $u \in U_{w_1}$ such that $\alpha(u)(x) = d$. We have $u \in [\![\Box\forall x p(x)]\!]^{s_1}_{\phi}$ iff $\langle w_2, \alpha(u)\rangle \in [\![\forall x p(x)]\!]^{s_1}_{\phi}$, but $(*)$ implies that $[\![\forall x p(x)]\!]^{s_1}_{\phi}$ is empty. This proves that for all $u \in U_{w_1}$ such that $\alpha(u)(x) = d$, $u \notin [\![\forall x \Box p(x) \Rightarrow \Box\forall x p(x)]\!]^{s_1}_{\phi}$.

As far as the converse implication is concerned, we consider instead the structure s_2 also outlined in Figure 2. In particular, we have $W = \{w_1, w_2, w_3\}$ and $D = \{d, d'\}$. We set

$$R_{\alpha(u)} = \{\langle w_2, w_3\rangle\}\,,\ \text{if } \alpha(u)(x) = d\,; \qquad R_{\alpha(u)} = \{\langle w_2, w_1\rangle\}\,,\ \text{otherwise}\,.$$

Define $[p]_w$ by

$$0 = [p]_{w_1}(d) = [p]_{w_1}(d') = [p]_{w_2}(d) = [p]_{w_2}(d')\,, \quad 1 = [p]_{w_3}(d) = [p]_{w_3}(d')$$

Thus, $[\![p(x)]\!]^{s_1}_{\phi} = [\![\forall x p(x)]\!]^{s_1}_{\phi} = U_{w_3}$. Consider any $u \in U_{w_2}$ such that $\alpha(u)(x) = d$; the equality $[\![\forall x p(x)]\!]^{s_1}_{\phi} = U_{w_3}$ implies $u \in [\![\Box\forall x p(x)]\!]^{s_1}_{\phi}$ because $R_{\alpha(u)} = \{\langle w_2, w_3\rangle\}$. Let u' be any element of U_{w_2} such that $\alpha(u') = \alpha(u)^x_{d'}$; then $R_{\alpha(u')} = \{\langle w_2, w_1\rangle\}$ and hence $u' \notin [\![\Box p(x)]\!]^{s_1}_{\phi}$. Moreover, $\alpha(u) \stackrel{x}{=} \alpha(u')$ and hence $u \notin [\![\forall x \Box p(x)]\!]^{s_1}_{\phi}$. We can then conclude $u \notin [\![\Box\forall x p(x) \Rightarrow \forall x \Box p(x)]\!]^{s_1}_{\phi}$. $\triangle$

Remark 4.3 *Quantifying bound variables.* The implication $\Box\forall x\varphi \Rightarrow \forall x\Box\varphi$ is often considered a theorem of modal first-order logic (see, for instance, [11]). Of course, there must be some step in the proof of this formula which is not allowed in our semantics. The crucial passage, in fact, is the use of (equivalents of) the first-order validity $\psi \Rightarrow \forall x\psi$, for x not free in ψ. The structure s_2 of Figure 2 can be used to show that this first-order validity can be falsified in our semantics.

Assume that p is a 0-ary relation symbol, so that $[p]_w = 0$ or $[p]_w = 1$, and assume $[p]_w = 1$ only for $w = w_3$. According to the example above, we have $u \in [\![\Box p]\!]^{s_2}_{\phi}$ for every $u \in U_{w_2}$ such that $\alpha(u)(x) = d$. However, as the example above shows, for these u's we also have $u \notin [\![\forall x \Box p]\!]^{s_2}_{\phi}$.

It can be easily verified, however, that $\psi \Rightarrow \forall x\psi$ is a validity in our semantics if (x has no free occurrences in ψ and) ψ does not contain modal operators. This means that no genuine first-order validity is lost.

This turns out to be a key feature of our approach to modal quantifier logic: $\Box\forall x\psi$ is not necessarily equivalent to $\forall x\Box\forall x\psi$. This might seem strange but it agrees with our point of view of looking at quantifiers as modalities (over assignments). Indeed, in general, $\Box_1\Box_2\psi$ is not equivalent to $\Box_2\Box_1\Box_2\psi$. Furthermore, it opens the possibility of not having the Barcan formula as explained in Example 4.2. $\triangle$

Although our main goal is the study of fibring, our semantic approach did lead us to a novel semantics for modal first-order logic as presented in Example 4.1. The striking novelty in our approach is the fact that the accessibility relation may depend on the assignments (but, as a particular case, we can impose $R_\delta = R_{\delta'}$ for all δ, δ' in Δ as described below). Although it may seem strange, having the relation depending on the assignment is the natural thing to do from a fibring point of view. Indeed, from this point of view, a modal quantifier logic is just a a bi-dimensional modal logic with one dimension dedicated to the proper modality and the other dimension dedicated to the quantifiers seen as modalities over assignments. Therefore, a model is a cloud of points like in Figure 1 and at each point we have a (horizontal) modal line and a (vertical) quantifier line. More precisely, at each point we have a (horizontal) modal structure and a (vertical) fol structure. At two different points of the same vertical line we may of course have two different modal structures and, hence, two different accessibility relations.

The assignment-dependent accessibility relation is the key ingredient of our approach towards obtaining a modal quantifier logic without the Barcan formulae. This desideratum was already achieved by other means in other approaches to modal quantification, like [10, 12], where, however, key ingredients are flexible domains of individuals and existence properties. On the contrary, our semantics of modal quantifier logic (in Example 4.1) uses rigid domains.

Definability

Although definability issues are beyond the scope of this paper, we briefly address the problem in order to explain better the semantics we proposed for modal first-order logic. Namely, we present the appropriate notion of definability and look at the definability of properties related to Barcan formulae.

We already observed that, in our semantics, the accessibility relation between possible worlds may vary with the assignment at hand. The usual possible worlds Kripke semantics can then be viewed as a particular case of the one we considered, in which $R_\delta = R_{\delta'}$ for all δ, δ' in Δ. If this holds in a model m, we will say that m is *standard*. In a given standard modal model m, we have that, for every formula φ and every variable x, $[\![\Diamond\varphi \Rightarrow \forall x \Diamond \exists x \varphi]\!]^m = U$. In general, the converse implication is not true, however: the function $[\cdot]$ can be chosen suitably, in a way such that $[\![\Diamond\varphi \Rightarrow \forall x \Diamond \exists x \varphi]\!]^m = U$ for every formula φ, even if $R_\delta \neq R_{\delta'}$ for some δ, δ' in Δ. In order to be able to define the class of classical modal structures, we need to consider a different notion of definability.

Definition 4.4 Given a modal model $m = \langle D, W, R, V, \mathcal{I}\rangle$ with corresponding structure $s = \langle U, \Delta, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [\cdot]\rangle$, the *frame* $\mathcal{F}(m)$ of m is the tuple $\langle U, \Delta, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, R\rangle$. We say that model m' is a *frame variant* of model m if $\mathcal{F}(m) = \mathcal{F}(m')$.

Lemma 4.5 Assume that the language L contains an atomic formula $\varphi = p(x_1, \ldots, x_k)$ $(k \geq 0)$ which is neither an equality nor an inequality. Then the modal model m is standard iff, for every frame variant m' of m, $[\![\Diamond\varphi \Rightarrow \forall x \Diamond \exists x \varphi]\!]^{m'} = U'$.

Proof: We already observe that the implication $\Diamond\varphi \Rightarrow \forall x \Diamond \exists x \varphi$ is true for every φ in every standard model and hence in every frame variant of it.

In order to prove the converse implication, we consider the particular case in which k=1 and write x for x_1; the general case can be proved in a quite similar way. We assume that, for every frame variant m' of m, $[\![\Diamond p(x) \Rightarrow \forall x \Diamond \exists x p(x)]\!]^{m'} = U'$. Assume, as a *reductio*, that for some $w, w' \in W$ and $\delta, \delta' \in \Delta$, $wR_\delta w'$ and not-$wR_{\delta'}w'$. Consider a particular frame variant m' of m in which $[p]_v(d) = 1$ iff $v = w'$ and $d = \delta(x)$, so that $\emptyset \neq [\![p(x)]\!]^{s'} \subseteq U_{w'}$, $[\![\exists x p(x)]\!]^{s'} = U_{w'}$, and $\langle w, \delta \rangle \in [\![\Diamond p(x)]\!]^{s'}$. Since w is not $R_{\delta'}$ related to w', $\langle w, \delta' \rangle \notin [\![\Diamond \exists x p(x)]\!]^{s'}$ and hence $\langle w, \delta \rangle \notin [\![\forall x \Diamond \exists x p(x)]\!]^{s'}$. QED

It is straightforward to prove that, for every standard modal model m and every formula φ, $[\![\forall x \Box \varphi \Rightarrow \Box \forall x \varphi]\!]^m = U$ and $[\![\Box \forall x \varphi \Rightarrow \forall x \Box \varphi]\!]^m = U$. The problem of the class of models which can be defined, in the sense of Lemma 4.5, by means of these two formulae is rather complex and it lies beyond the scope of the present paper. The following proposition provides an example of a property of the relations R_δ which follows from the Barcan formula.

Proposition 4.6 Let m be a modal model such that

$$[\![\forall x \Box p(x) \Rightarrow \Box \forall x p(x)]\!]^{m'} = U'$$

for every frame variant m' of m, and let w_1, w_2 be two possible worlds in m such that $w_1 R_{\delta_0} w_2$ for some $\delta_0 \in \Delta$. Then, for every $d \in D$, there exists a $\delta \in \Delta$ such that $[x]_\delta = d$ and $w_1 R_\delta w_2$.

Proof: Define the set D' by:

$$D' = \{d \in D : \exists \delta{\in}\Delta \ [x]_\delta = d \text{ and } w_1 R_\delta w_2\}$$

We prove that $D' = D$. Consider a frame variant m' of m in which $[p]_w = D$ for all $w \neq w_2$ and $[p]_{w_2} = D'$. We first show that $[\![\Box p(x)]\!]^{m'} \supseteq U_{w_1}$. Given any $\langle w_1, \delta \rangle$, if $w \neq w_2$ then $w_1 R_\delta w$ implies trivially $\langle w, \delta \rangle \in [\![p(x)]\!]^{m'}$. If $w = w_2$, then $w_1 R_\delta w$ implies $[x]_\delta \in D'$ and hence $\langle w, \delta \rangle \in [\![p(x)]\!]^{m'}$. The inclusion $[\![\Box p(x)]\!]^{m'} \supseteq U_{w_1}$ implies $[\![\forall x \Box p(x)]\!]^{m'} \supseteq U_{w_1}$, which implies, by the Barcan formula, $[\![\Box \forall x p(x)]\!]^{m'} \supseteq U_{w_1}$. The assumption $w_1 R_{\delta_0} w_2$ yields $\langle w_2, \delta_0 \rangle \in [\![\forall x p(x)]\!]^{m'}$, which implies $\langle w_2, \delta \rangle \in [\![p(x)]\!]^{m'}$ for all $\delta \in \Delta$ and hence $D' = D$. QED

5 Hilbert calculi

We now turn our attention to the deductive component of a fob logic. As in [17, 14], we adopt a Hilbert style for this component. However, the problem is now much more complex because rules in fob logics frequently have side constraints like "provided that a term is free for a variable in a formula". Such constraints correspond to the following abstractions (adapted from [15]):

Definition 5.1 (i) A Σ-*proviso* is a map from $Sub(\Sigma)$ to $\{0, 1\}$. (ii) A *proviso* π is a family $\{\pi_\Sigma\}_{\Sigma \in \mathrm{fobSig}}$, where fobSig is the class of all fob signatures and each π_Σ is a Σ-proviso, such that $\pi_{\Sigma'}(\rho) = \pi_\Sigma(\rho)$ for every Σ-substitution ρ whenever $\Sigma' \supseteq \Sigma$. $\triangle$

Intuitively, we have $\pi_\Sigma(\rho) = 1$ iff the Σ-substitution ρ is allowed. Provisos are well known in first-order logic. For example, we can say that any substitution instance $\xi\rho \Rightarrow \forall x\xi\rho$ of $\xi \Rightarrow \forall x\xi$ is a validity of first-order logic *provided that* x is not free in $\xi\rho$; in this case, we have $\pi(\rho) = 1$ iff x is not free in $\xi\rho$. Similarly, the restriction "$\theta\rho$ is free for variable x in formula $\xi\rho$"for the applicability of the first-order axiom $\forall x\xi \Rightarrow \xi^x_\theta$ can be expressed by means of a suitable proviso.

The definition of proviso as a family of functions indexed by signatures wants to express that, in general, the constraints which appear in deduction rules are actually families of constraints, depending on the language we are considering. The constraint "x not free in $\xi\rho$", for instance, has the same meaning in all first-order languages, but, as a function defined on the possible substitutions depends on the language we are considering.

The *unit proviso* **1** maps at each signature Σ every Σ-substitution to 1. And the *zero proviso* **0** maps at each signature Σ every Σ-substitution to 0.

The provisos $\mathrm{cfo}(\xi)$ and $\mathrm{rig}(\xi)$, defined as follows, will be used frequently throughout the paper. For each signature Σ:

- $\mathrm{cfo}_\Sigma(\xi)(\rho) = 1$ iff the formula $\xi\rho$ is a closed first-order formula;
- $\mathrm{rig}_\Sigma(\xi)(\rho) = 1$ iff the formula $\xi\rho$ is an equality or inequality of rigid terms, i.e. terms in $X \cup I$.

We denote the sets of all Σ-provisos and all provisos by *Prov*(Σ) and *Prov*, respectively. Given a proviso π we say that π_Σ is the Σ-*instance* of π. When no confusion arises we may write $\pi(\rho)$ for $\pi_\Sigma(\rho)$.

Using the notion of proviso we are ready to define precisely what we mean by a fob inference rule for some signature Σ.

Definition 5.2 A Σ-*rule* is a triple $\langle\Psi, \eta, \pi\rangle$ where:

- $\Psi \subseteq L(\Sigma, X, \Theta, \Xi)$ is finite (the set of *premises*);
- $\eta \in L(\Sigma, X, \Theta, \Xi)$ (the *conclusion*);
- $\pi \in$ *Prov* (the *constraint*). $\triangle$

One can reasonably find strange that, in the previous definition, the last component of a Σ-rule is not an element of *Prov*(Σ), but a whole family π. This fact has technical reasons; namely, we want to be able to consider a Σ-rule also as a Σ'-rule, where Σ' is a richer signature. In this case, we need to know how the proviso works on Σ'-substitutions.

It is worth observing that we loose no generality by endowing a rule with just one proviso. Indeed, although rules may be stated in practice with a collection of constraints, it is straightforward to represent any such collection Π of provisos by the "product" proviso that at each signature Σ maps each Σ-substitution ρ to 1 iff all the elements of Π at Σ do so for ρ. In the sequel, given two Σ-provisos π and π' we denote their product by $\pi * \pi'$.

It is now natural to introduce the notion of fob Hilbert system as a collection of rules. This collection must be given a inner structure because, as already done in [14, 17], we want to distinguish between proof rules and derivation rules, and, in the context of modal fob logics, we want to distinguish between proof rules for quantifiers and proof rules for modalities.

Definition 5.3 A Σ-*Hilbert system* is a tuple $\langle R_{\mathrm{d}}, R_{\mathrm{Qp}}, R_{\mathrm{Op}}, R_{\mathrm{p}}\rangle$ where:

- R_{d} is a set of Σ-rules (the *derivation rules*);
- $R_{\mathrm{Qp}} \supseteq R_{\mathrm{d}}$ is a set of Σ-rules (the *quantifier proof rules*);
- $R_{\mathrm{Op}} \supseteq R_{\mathrm{d}}$ is a set of Σ-rules (the *modal proof rules*);
- $R_{\mathrm{p}} \supseteq R_{\mathrm{Qp}} \cup R_{\mathrm{Op}}$ is a set of Σ-rules (the *proof rules*). $\triangle$

The distinction between proof and derivation rules is understood in terms of two semantic entailments introduced in Definition 3.5.

Definition 5.4 (i) Let s be a Σ-structure and $H = \langle R_{\mathrm{d}}, R_{\mathrm{Qp}}, R_{\mathrm{Op}}, R_{\mathrm{p}}\rangle$ be a Σ-Hilbert system such that, for every Σ-substitution ρ:

- for every $\langle \Psi, \eta, \pi\rangle \in R_{\mathrm{p}}$, $s \Vdash^{\Sigma}_{\mathrm{p}} \eta\rho$ whenever $s \Vdash^{\Sigma}_{\mathrm{p}} \psi\rho$ for every $\psi \in \Psi$ and $\pi(\rho) = 1$;
- for every $\langle \Psi, \eta, \pi\rangle \in R_{\mathrm{d}}$ and $u \in U$, $su \Vdash^{\Sigma}_{\mathrm{d}} \eta\rho$ whenever $su \Vdash^{\Sigma}_{\mathrm{d}} \psi\rho$ for every $\psi \in \Psi$ and $\pi(\rho) = 1$.

Then, s is said to be *appropriate for H*.

(ii) If $A(m)$ is appropriate for the Hilbert system H for every model m in the Σ-interpretation system $\langle M, A\rangle$, then H is said to be *sound* for $\langle M, A\rangle$. $\triangle$

The distinction between quantifier and modal proof rules will be used only at the proof-theoretic level. We delay its justification until we address later on in this section the problem of defining precisely what we mean by a vertically and a horizontally persistent logic. But to this end we need first to introduce the notion of Q-proof and O-proof that we shall do in the context of a Hilbert framework (the proof-theoretic counterpart of interpretation framework).

Definition 5.5 A *Hilbert framework* is a pair $\mathcal{H} = \langle Sig, H\rangle$ where:

- *Sig* is a class of fob signatures;
- H maps each $\Sigma \in Sig$ to a Σ-Hilbert system. $\triangle$

Logics are often endowed with uniform Hilbert calculi in the sense that their rules do not depend on the signature at hand. More precisely:

Definition 5.6 A Hilbert framework $\mathcal{H}$ is said to be *uniform* iff:

1. $H(\Sigma) = H(\Sigma')$ for every $\Sigma, \Sigma' \in Sig$;

2. for every signature $\Sigma \in Sig$ and proof rule $\langle \Psi, \eta, \pi \rangle$ in $H(\Sigma)$, $\pi_\Sigma(\rho) = \pi_\Sigma(\rho')$, where, for each $\theta \in \Theta$ and $\xi \in \Xi$, $\rho'(\theta)$ and $\rho'(\xi)$ are respectively obtained from $\rho(\theta)$ and $\rho(\xi)$ by replacing some occurrences of i by x, provided that x is fresh in $\Psi\rho \cup \{\eta\rho\}$. $\triangle$

Uniform Hilbert frameworks are common in logic. When, for instance, we say that $\alpha \Rightarrow (\beta \Rightarrow \alpha)$ is an axiom of first-order logic, we mean that, in every language, every instance of $\alpha \Rightarrow (\beta \Rightarrow \alpha)$ in that language is an axiom of the version of first-order logic that is based on that language.

Clause 2. above is expected in a signature-independent framework since individual symbols belong to the signature. However, it may happen that a logic has some individual symbols that are present in all signatures. Even in this case, Clause 2. imposes that provisos should be blind to them. This additional requirement is nevertheless fulfilled by the Hilbert calculi given to usual logics.

As an illustration of the concepts above, consider the following Hilbert framework for the modal K first-order logic whose semantics was presented in Example 4.1.

Example 5.7 *Modal K first-order logic - Hilbert framework.*

$$\mathcal{H}_{\mathrm{KFOL}} = \langle Sig, H \rangle$$

(i) The class *Sig* is as introduced in Example 4.1. Note that we shall also use other connectives defined as abbreviations in the standard way.

(ii) Each Hilbert system $H(\Sigma(I,F,P)) = \langle R_{\mathrm{d}}, R_{\mathrm{Qp}}, R_{\mathrm{Op}}, R_{\mathrm{p}} \rangle$ is as follows, where possibly indexed θ and ξ range respectively over Θ, the set of term schema variables, and over Ξ, the set of formula schema variables.

- R_{d} is composed of the following rules:
  - $\langle \emptyset, \varphi, \mathbf{1} \rangle$ for every tautological schema formula φ;
  - $\langle \emptyset, \theta = \theta, \mathbf{1} \rangle$;
  - $\langle \emptyset, \theta_1 = \theta_2 \Rightarrow \theta_2 = \theta_1, \mathbf{1} \rangle$;
  - $\langle \emptyset, \theta_1 = \theta_2 \Rightarrow (\theta_2 = \theta_3 \Rightarrow \theta_1 = \theta_3), \mathbf{1} \rangle$;
  - $\langle \emptyset, \theta_1 = \theta_1' \Rightarrow (\ldots \Rightarrow (\theta_k = \theta_k' \Rightarrow \theta^{x_1,\ldots,x_k}_{\theta_1,\ldots,\theta_k} = \theta^{x_1,\ldots,x_k}_{\theta_1',\ldots,\theta_k'}) \ldots), \mathbf{1} \rangle$;
  - $\langle \emptyset, \theta_1 = \theta_1' \Rightarrow (\ldots \Rightarrow (\theta_k = \theta_k' \Rightarrow (\xi^{x_1,\ldots,x_k}_{\theta_1,\ldots,\theta_k} \Rightarrow \xi^{x_1,\ldots,x_k}_{\theta_1',\ldots,\theta_k'})) \ldots), \mathrm{atm}(\xi) \rangle$ where $\mathrm{atm}(\xi) = 1$ iff ξ is atomic;
  - $\langle \emptyset, \theta_1 \neq \theta_2 \Leftrightarrow (\neg\, \theta_1 = \theta_2), \mathbf{1} \rangle$;
  - $\langle \{\xi_1, \xi_1 \Rightarrow \xi_2\}, \xi_2, \mathbf{1} \rangle$.

- R_{Qp} is composed of the rules in R_{d} and the following proper quantification rules:
  - $\langle \emptyset, (\forall x\, (\xi_1 \Rightarrow \xi_2)) \Rightarrow (\forall x\, \xi_1 \Rightarrow \forall x\, \xi_2), \mathbf{1} \rangle$;

- $\langle \emptyset, \xi \Rightarrow \forall x\, \xi, x\not\in\xi \rangle$ where $x\not\in\xi\,(\rho) = 1$ iff x does not occur free in $\xi\rho$ and $\xi\rho$ does not contain modalities;
- $\langle \emptyset, (\forall x\, \xi) \Rightarrow \xi^x_\theta, \theta \triangleright x{:}\xi \rangle$ where $\theta \triangleright x{:}\xi\,(\rho) = 1$ iff when replacing the free occurrences of x in $\xi\rho$ no variable in $\theta\rho$ is captured by a quantifier and no non-rigid replacement is made within the scope of a modality;
- $\langle \{\xi\}, \forall x\, \xi, \mathbf{1} \rangle$.

- R_{Op} is composed of the rules in R_{d} and the following proper modal rules:
  - $\langle \emptyset, (\Box\,(\xi_1 \Rightarrow \xi_2)) \Rightarrow (\Box\,\xi_1 \Rightarrow \Box\,\xi_2), \mathbf{1} \rangle$;
  - $\langle \emptyset, \xi \Rightarrow \Box\,\xi, \mathrm{rig}(\xi) \rangle$;
  - $\langle \{\xi\}, \Box\,\xi, \mathbf{1} \rangle$.
- $R_{\mathrm{p}} = R_{\mathrm{Qp}} \cup R_{\mathrm{Op}}$.

Observe that this Hilbert framework is uniform. Furthermore, it is easy to check that, for each signature $\Sigma(I, F, P)$, the Hilbert system $H(\Sigma(I, F, P))$ is sound for the interpretation system $S(\Sigma(I, F, P))$ introduced in Example 4.1.

Note also that we find again, now at the deduction level, the feature of our approach discussed in Remark 4.3: $\Box\forall x\psi$ is not necessarily equivalent to $\forall x\Box\forall x\psi$. This is a direct consequence of the strong proviso for axiom $\xi \Rightarrow \forall x\, \xi$. We must stress that without that proviso the deductive system would be unsound with respect to the interpretation framework given in Example 4.1: consider the counterexample in Remark 4.3.

The strong proviso for axiom $(\forall x\, \xi) \Rightarrow \xi^x_\theta$ is also essential. Without this proviso, given a flexible symbol s, we would be able to infer $(s = s) \Rightarrow \Diamond(s > s)$ from $\forall x((s = x) \Rightarrow \Diamond(s > x))$. Obviously, the latter is a satisfiable formula while the former is not. △

Before defining precisely the four notions of inference within the context of a Hilbert framework, we need to say what we mean by applying a schema Σ-substitution to a Σ-instance of a proviso. Given a proviso π and a schema Σ-substitution σ, we denote by $\pi_\Sigma\sigma$ the map such that: $(\pi_\Sigma\sigma)(\rho) = \pi_\Sigma(\sigma\rho)$. Obviously, $\mathbf{1}_\Sigma\sigma = \mathbf{1}_\Sigma$ and $\mathbf{0}_\Sigma\sigma = \mathbf{0}_\Sigma$. Furthermore, for every Σ-substitution ρ, we have that either $\pi_\Sigma\rho = \mathbf{1}_\Sigma$ or $\pi_\Sigma\rho = \mathbf{0}_\Sigma$, depending on whether $\pi_\Sigma(\rho) = 1$ or $\pi_\Sigma(\rho) = 0$, respectively.

Definition 5.8 Within the context of a Hilbert framework $\mathcal{H}$:
(i) A Σ-*proof* of $\varphi \in L(\Sigma, X, \Theta, \Xi)$ from $\Gamma \subseteq L(\Sigma, X, \Theta, \Xi)$ *constrained by* $\pi \in$ *Prov*(Σ) is a sequence $\langle \varphi_1, \pi_1 \rangle, \ldots, \langle \varphi_n, \pi_n \rangle$ of pairs in $L(\Sigma, X, \Theta, \Xi) \times$ *Prov*(Σ) such that:

- $\pi \neq \mathbf{0}_\Sigma$;
- φ is φ_n and π is π_n;
- for each $i = 1, \ldots, n$:

- either $\varphi_i \in \Gamma$ and $\pi_i = \mathbf{1}_\Sigma$;
- or there is a rule $r = \langle \Psi, \eta, \pi' \rangle \in R_{\rm p}$ within $H(\Sigma)$ and a schema Σ-substitution σ such that:
  - φ_i is $\eta\sigma$;
  - $\Psi\sigma = \{\varphi_{j_1}, \ldots, \varphi_{j_k}\} \subseteq \{\varphi_1, \ldots, \varphi_{i-1}\}$;
  - $\pi_i = \pi_{j_1} * \cdots * \pi_{j_k} * \pi'_\Sigma\sigma$.

In this case we write $\Gamma \vdash^\Sigma_{\rm p} \varphi : \pi$ or, simply, $\Gamma \vdash_{\rm p} \varphi : \pi$. If $\emptyset \vdash^\Sigma_{\rm p} \varphi : \pi$ we say that φ is a Σ-*theorem constrained by* π. If $\pi = \mathbf{1}_\Sigma$ we omit π since such a constraint is always fulfilled. In this case we are in the presence of a (non constrained) Σ-proof or a (non constrained) Σ-theorem.

(ii) For R $\in \{R_{\rm Qp}, R_{\rm Op}, R_{\rm d}\}$ a ΣR-*inference* of $\varphi \in L(\Sigma, X, \Theta, \Xi)$ from $\Gamma \subseteq L(\Sigma, X, \Theta, \Xi)$ *constrained by* $\pi \in Prov(\Sigma)$ is a sequence $\langle \varphi_1, \pi_1 \rangle, \ldots, \langle \varphi_n, \pi_n \rangle$ of pairs in $L(\Sigma, X, \Theta, \Xi) \times Prov(\Sigma)$ such that:

- $\pi \neq \mathbf{0}_\Sigma$;
- φ is φ_n and π is π_n;
- for each $i = 1, \ldots, n$:
  - either $\varphi_i \in \Gamma$ and $\pi_i = \mathbf{1}_\Sigma$;
  - or φ_i is a Σ-theorem constrained by π' and $\pi_i = \pi'$;
  - or there is a rule $r = \langle \Psi, \eta, \pi' \rangle \in$ R within $H(\Sigma)$ and a schema Σ-substitution σ such that:
    - φ_i is $\eta\sigma$;
    - $\Psi\sigma = \{\varphi_{j_1}, \ldots, \varphi_{j_k}\} \subseteq \{\varphi_1, \ldots, \varphi_{i-1}\}$;
    - $\pi_i = \pi_{j_1} * \cdots * \pi_{j_k} * \pi'_\Sigma\sigma$.

A ΣR-inference will be called a ΣQ-*proof*, or a ΣO-*proof*, or a Σ-*derivation*, in all cases *constrained by* π, according to whether R is $R_{\rm Qp}$, or $R_{\rm Op}$, or $R_{\rm d}$, respectively. Moreover, we will write $\Gamma \vdash^\Sigma_{\rm Qp} \varphi : \pi$, or $\Gamma \vdash^\Sigma_{\rm Op} \varphi : \pi$, or $\Gamma \vdash^\Sigma_{\rm d} \varphi : \pi$ with the obvious meaning. As above, we omit π in the unconstrained cases, that is, when π is the unit Σ-proviso. △

According to this definition, for R $\in \{R_{\rm Qp}, R_{\rm Op}, R_{\rm d}\}$, the difference between a ΣR-inference and a Σ-proof, from a given set Γ, is that, in the former, the elements of Γ can be involved as premises only in rules in R. The inclusion relationships between the various sets of rules imply that 1) if $\Gamma \vdash_{\rm d} \varphi : \pi$ then $\Gamma \vdash_{\rm Qp} \varphi : \pi$ and $\Gamma \vdash_{\rm Op} \varphi$, and 2) if $\Gamma \vdash_{\rm Qp} \varphi : \pi$ or $\Gamma \vdash_{\rm Op} \varphi : \pi$ then $\Gamma \vdash_{\rm p} \varphi : \pi$. Moreover, every Σ-theorem is ΣR-deducible for any R from any set Γ.

In the sequel, it will be often convenient to use the 'closure' notation for provability and derivability, that is, for $\vdash \in \{\vdash_{\rm p}, \vdash_{\rm Qp}, \vdash_{\rm Op}, \vdash_{\rm d}\}$ and any set Γ of formulae, we set

$$\Gamma^{\vdash} = \{\varphi : \Gamma \vdash \varphi\}$$

For R $\in \{R_{\mathrm{p}}, R_{\mathrm{Qp}}, R_{\mathrm{Op}}, R_{\mathrm{d}}\}$, the deduction sequence $\langle\varphi_1, \pi_1\rangle, \ldots, \langle\varphi_n, \pi_n\rangle$ of $\Gamma \vdash^{\Sigma}_{\mathrm{R}} \varphi : \pi$ is said to be *sober* iff no proper subsequence is a deduction sequence for $\Gamma \vdash^{\Sigma}_{\mathrm{R}} \varphi : \pi$. Obviously, from any deduction sequence we can always extract a sober one by removing superfluous steps.

If a deduction is done without using schema variables, the resulting π is necessarily $\mathbf{1}_\Sigma$. Another way of obtaining such non schematic results is by producing an instance of a schematic result by applying a substitution $\rho \in Sub(\Sigma)$ such that $\pi(\rho) = 1$. Concerning such non schematic deductions, the following result is easily established by induction.

Proposition 5.9 For R $\in \{R_{\mathrm{p}}, R_{\mathrm{Qp}}, R_{\mathrm{Op}}, R_{\mathrm{d}}\}$ and $\rho \in Sub(\Sigma)$, if $\Gamma \vdash^{\Sigma}_{\mathrm{R}} \varphi : \pi$ with sober deduction sequence $\langle\varphi_1, \pi_1\rangle, \ldots, \langle\varphi_n, \pi_n\rangle$ and $\pi_\Sigma(\rho) = 1$, then we have $\Gamma\rho \vdash^{\Sigma}_{\mathrm{R}} \varphi\rho$ with deduction sequence $\langle\varphi_1\rho, \mathbf{1}_\Sigma\rangle, \ldots, \langle\varphi_n\rho, \mathbf{1}_\Sigma\rangle$.

The next result will also be used later on. In short, it states sufficient conditions for replacing individual symbols by variables in a derivation.

Proposition 5.10 Assume that, in a uniform Hilbert framework H, $\Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi : \pi$ and let x be any variable fresh in the corresponding derivation. Then, for every i which does not occur in the rules of $H(\Sigma)$, $\Gamma^i_x \vdash^{\Sigma}_{\mathrm{d}} \varphi^i_x : \pi$.

Proof: Observe first that, given any rule $r = \langle\Psi, \eta, \pi\rangle$, any i which does not occur in Ψ, any fresh x, and any schema Σ-substitution σ, if the set $\{\varphi_1, \ldots, \varphi_k\}$ is contained in $\Psi\sigma$, then the set $\{(\varphi_1)^i_x, \ldots, (\varphi_k)^i_x\}$ is contained in $\Psi\sigma^i_x$. This means that, if ψ is a theorem, then its proof can be turned to a proof of ψ^i_x by replacing every schema substitution σ involved in the proof by σ^i_x. Moreover, it is trivial that $\psi \in \Gamma$ implies $\psi^i_x \in \Gamma^i_x$. Then, the substitutions $\sigma \to \sigma^i_x$ transform a derivation of φ from Γ into a derivation of φ^i_x from Γ^i_x. As far as the provisos are concerned, since H is assumed to be uniform, $\pi_\Sigma\sigma^i_x(\rho) = \pi_\Sigma(\sigma^i_x\rho) = \pi_\Sigma(\sigma\rho) = \pi_\Sigma\sigma(\rho)$ for every ρ and hence $\pi_\Sigma\sigma^i_x = \pi_\Sigma\sigma$. This shows that the substitutions $\sigma \to \sigma^i_x$ do not change the constraints in the derivation. QED

We now proceed to identify interesting classes of Hilbert frameworks. When proving the completeness theorem in the next section we need to assume that we are working with frameworks in these classes. This assumption is not too restrictive since fob logics tend to be endowed with such frameworks.

We start by defining vertically and horizontally persistent frameworks. These notions shed some light on the need for the distinction between quantifier and modal proof rules.

But first we introduce some useful Σ-provisos. Given a set Ψ of schema formulae over Σ:

- $\mathrm{cfo}_\Sigma(\Psi) = \lambda\rho. \bigwedge_{\psi\in\Psi} \mathrm{cfo}_\Sigma(\xi)[\xi/\psi](\rho)$;
- $\mathrm{rig}_\Sigma(\Psi) = \lambda\rho. \bigwedge_{\psi\in\Psi} \mathrm{rig}_\Sigma(\xi)[\xi/\psi](\rho)$;

where $[\xi/\psi]$ denotes the Σ-schema substitution that replaces ξ by ψ.

Definition 5.11 We say that a Hilbert framework $\mathcal{H}$ is *vertically persistent* and *horizontally persistent* iff, respectively, the following properties (VP) and (HP) hold for every signature $\Sigma \in \mathit{Sig}$ and Γ, Ψ, φ in $L(\Sigma, X, \Theta, \Xi)$:

$$\textbf{(VP)}\ \frac{\Gamma^{\vdash_{\mathrm{P}}}, \Psi \vdash^{\Sigma}_{\mathrm{Qp}} \varphi : \pi * \mathrm{cfo}_{\Sigma}(\Psi)}{\Gamma^{\vdash_{\mathrm{P}}}, \Psi \vdash^{\Sigma}_{\mathrm{d}} \varphi : \pi * \mathrm{cfo}_{\Sigma}(\Psi)};$$

$$\textbf{(HP)}\ \frac{\Gamma^{\vdash_{\mathrm{P}}}, \Psi \vdash^{\Sigma}_{\mathrm{Op}} \varphi : \pi * \mathrm{rig}_{\Sigma}(\Psi)}{\Gamma^{\vdash_{\mathrm{P}}}, \Psi \vdash^{\Sigma}_{\mathrm{d}} \varphi : \pi * \mathrm{rig}_{\Sigma}(\Psi)}.$$

We say that $\mathcal{H}$ is *persistent* iff it is both vertically and horizontally persistent.
$\triangle$

Intuitively, in a persistent framework, whatever we can Qp-prove from a set of closed first-order formulae, we can also derive from the same set; and whatever we can Op-prove from a set of rigid formulae we can derive from the same set. That is, quantifier proof rules do not bring anything new from a set of closed first-order formulae and modal proof rules do not bring anything new from a set of rigid formulae.

The distinction between quantifier and modal proof rules also plays an essential role in the notion of congruent framework.

Definition 5.12 A Hilbert framework $\mathcal{H}$ is said to be *congruent* iff for every signature $\Sigma \in \mathit{Sig}$:

1. for every Qp-deductively closed $\Gamma' \subseteq L(\Sigma, X, \Theta, \Xi)$, Op-deductively closed $\Gamma'' \subseteq L(\Sigma, X, \Theta, \Xi)$, $\varphi_1, \varphi'_1, \dots, \varphi_k, \varphi'_k$ in $L(\Sigma, X, \Theta, \Xi)$, and $c \in C_k$,

$$\frac{\Gamma', \Gamma'', \varphi_i \vdash^{\Sigma}_{\mathrm{d}} \varphi'_i : \pi \text{ and } \Gamma', \Gamma'', \varphi'_i \vdash^{\Sigma}_{\mathrm{d}} \varphi_i : \pi \quad \text{for } i = 1, \dots, k}{\Gamma', \Gamma'', c(\varphi_1, \dots, \varphi_k) \vdash^{\Sigma}_{\mathrm{d}} c(\varphi'_1, \dots, \varphi'_k) : \pi}$$

2. for every Qp-deductively closed $\Gamma \subseteq L(\Sigma, X, \Theta, \Xi)$, $\varphi_1, \varphi'_1, \dots, \varphi_k, \varphi'_k$ in $L(\Sigma, X, \Theta, \Xi)$, $q \in Q_k$ and $x \in X$,

$$\frac{\Gamma, \varphi_i \vdash^{\Sigma}_{\mathrm{d}} \varphi'_i : \pi \text{ and } \Gamma, \varphi'_i \vdash^{\Sigma}_{\mathrm{d}} \varphi_i : \pi \quad \text{for } i = 1, \dots, k}{\Gamma, qx\,(\varphi_1, \dots, \varphi_k) \vdash^{\Sigma}_{\mathrm{d}} qx\,(\varphi'_1, \dots, \varphi'_k) : \pi}$$

3. for every Op-deductively closed $\Gamma \subseteq L(\Sigma, X, \Theta, \Xi)$, $\varphi_1, \varphi'_1, \dots, \varphi_k, \varphi'_k$ in $L(\Sigma, X, \Theta, \Xi)$, and $o \in O_k$,

$$\frac{\Gamma, \varphi_i \vdash^{\Sigma}_{\mathrm{d}} \varphi'_i : \pi \text{ and } \Gamma, \varphi'_i \vdash^{\Sigma}_{\mathrm{d}} \varphi_i : \pi \quad \text{for } i = 1, \dots, k}{\Gamma, o(\varphi_1, \dots, \varphi_k) \vdash^{\Sigma}_{\mathrm{d}} o\,(\varphi'_1, \dots, \varphi'_k) : \pi}$$

$\triangle$

It is easy to understand why the set is required to be Qp-deductively closed or Op-deductively closed. Observe that, in first-order logic, for $\Gamma = \{\varphi, \psi\}$ we have $\Gamma, \varphi \vdash_{\mathrm{d}} \psi$ and $\Gamma, \psi \vdash_{\mathrm{d}} \varphi$, but in general we do not have $\Gamma, \forall x\, \varphi \vdash_{\mathrm{d}} \forall x\, \psi$. And, in modal logic, for $\Gamma = \{\varphi, \psi\}$ we have $\Gamma, \varphi \vdash_{\mathrm{d}} \psi$ and $\Gamma, \psi \vdash_{\mathrm{d}} \varphi$, but in general we do not have $\Gamma, \Box\, \varphi \vdash_{\mathrm{d}} \Box\, \psi$.

Finally, we consider the classes of Hilbert frameworks for equality and inequality. Recall that the symbols $=$ and $\neq$ are assumed to be always available in every fob logic. Their semantics is fixed (recall Definition 3.1), but, so far, we have not assumed at the proof-theoretic level anything about them.

Definition 5.13 A Hilbert framework $\mathcal{H}$ is said to be *for equality* iff, for every signature $\Sigma \in \mathit{Sig}$, Γ, φ in $L(\Sigma, X, \Theta, \Xi)$ and $t, t_1, t'_1, \ldots, t_k, t'_k$ in $T(\Sigma, X, \Theta)$:

1. $\vdash^{\Sigma}_{\mathrm{d}} t = t$;
2. $t_1 = t_2 \vdash^{\Sigma}_{\mathrm{d}} t_2 = t_1$;
3. $t_1 = t_2, t_2 = t_3 \vdash^{\Sigma}_{\mathrm{d}} t_1 = t_3$;
4. (i) $\dfrac{\Gamma \vdash^{\Sigma}_{\mathrm{d}} t_i = t'_i : \pi \text{ for } i = 1, \ldots, k}{\Gamma \vdash^{\Sigma}_{\mathrm{d}} f(t_1, \ldots, t_k) = f(t'_1, \ldots, t'_k) : \pi}$;

   (ii) $\dfrac{\Gamma \vdash^{\Sigma}_{\mathrm{d}} t_i = t'_i : \pi \text{ for } i = 1, \ldots, k}{\Gamma, p(t_1, \ldots, t_k) \vdash^{\Sigma}_{\mathrm{d}} p(t'_1, \ldots, t'_k) : \pi}$;
5. $\dfrac{\Gamma, t = i \vdash^{\Sigma}_{\mathrm{d}} \varphi : \pi}{\Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi : \pi}$, where i does not occur in the rules of $H(\Sigma)$ and $\pi(\rho) = 0$ whenever i occurs in $\Gamma\rho$ or in $\varphi\rho$. △

Clauses 1-4 impose that equality is a congruence relation. Clause 5 expresses a well known derived rule in ordinary first-order logic with equality that is reasonable to assume of any fob logic for equality.

Definition 5.14 A Hilbert framework $\mathcal{H}$ for equality is said to be *for inequality* iff, for every signature $\Sigma \in \mathit{Sig}$, Γ, φ in $L(\Sigma, X, \Theta, \Xi)$ and t_1, t_2 in $T(\Sigma, X, \Theta)$:

1. $\dfrac{\Gamma \vdash^{\Sigma}_{\mathrm{d}} t_1 = t_2 : \pi \text{ and } \Gamma \vdash^{\Sigma}_{\mathrm{d}} t_1 \neq t_2 : \pi}{\Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi : \pi}$;
2. $\dfrac{\Gamma, t_1 = t_2 \vdash^{\Sigma}_{\mathrm{d}} \varphi : \pi \text{ and } \Gamma, t_1 \neq t_2 \vdash^{\Sigma}_{\mathrm{d}} \varphi : \pi}{\Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi : \pi}$. △

Clauses 1 and 2 relate inequality with equality as expected when nothing is assumed about the available connectives.

We conclude this section by introducing the notion of logic as composed by an interpretation framework and a Hilbert framework, and by making precise the notions of completeness.

Definition 5.15 A *logic* is a triple $\mathcal{L} = \langle Sig, S, H\rangle$ such that $\langle Sig, S\rangle$ is an interpretation framework, $\langle Sig, H\rangle$ is a Hilbert framework and $H(\Sigma)$ is sound for $S(\Sigma)$ for every $\Sigma \in Sig$. △

Observe that we decided to deal only with logics with sound rules. This option is quite natural given the little interest of unsound rules, but it was motivated by technical reasons: working only with sound rules simplifies the treatment of fibring in Section 7.

As expected, a logic $\mathcal{L}$ is said to be for equality or inequality, or congruent, or (vertically or horizontally) persistent iff so is the underlying Hilbert framework.

Definition 5.16 A logic $\mathcal{L}$ is said to be (strongly):

- p-*sound* iff, for every $\Sigma \in Sig$, $\Gamma \subseteq L(\Sigma, X)$ and $\varphi \in L(\Sigma, X)$, $\Gamma \vDash^{\Sigma}_{\mathrm{p}} \varphi$ whenever $\Gamma \vdash^{\Sigma}_{\mathrm{p}} \varphi$;
- d-*sound* iff, for every such Σ, Γ, φ, $\Gamma \vDash^{\Sigma}_{\mathrm{d}} \varphi$ whenever $\Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi$;
- p-*complete* iff, for every such Σ, Γ, φ, $\Gamma \vdash^{\Sigma}_{\mathrm{p}} \varphi$ whenever $\Gamma \vDash^{\Sigma}_{\mathrm{p}} \varphi$;
- d-*complete* iff, for every such Σ, Γ, φ, $\Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi$ whenever $\Gamma \vDash^{\Sigma}_{\mathrm{d}} \varphi$.

The logic $\mathcal{L}$ will be said to be *sound* iff it is p-*sound* and d-*sound* and *complete* iff it is p-*complete* and d-*complete*. △

Note that soundness and completeness are stated only for non schematic formulae in $L(\Sigma, X)$. Indeed, it would be impossible to consider those notions for schema formulae in $L(\Sigma, X, \Theta, \Xi)$ since there is no semantic counterpart to provisos.

Clearly, a logic $\mathcal{L}$ is sound iff, for each signature $\Sigma \in Sig$, the Hilbert system $H(\Sigma)$ is sound for the Σ-interpretation system $S(\Sigma)$. Therefore, (according to Definition 5.15) every logic is sound.

Example 5.17 *Modal K first-order logic.* Consider the logic composed of the interpretation framework introduced in Example 4.1 and the Hilbert framework defined in Example 5.7. This logic is for equality and inequality. Furthermore, the Qp-rule $\langle \emptyset, \xi \Rightarrow \forall x\, \xi, \{x \not\in \xi\}\rangle$ ensures vertical persistency, and the Op-rule $\langle \emptyset, \xi \Rightarrow \Box\, \xi, \mathrm{rig}(\xi)\rangle$ guarantees horizontal persistency. In order to prove that the logic is congruent, we can use classical results from first-order and modal logic; it is worth observing that Clause 1 in Definition 5.12 holds for all sets Γ' and Γ''. It is also straightforward to verify that the logic we are considering is sound, taking into account the previous observation. △

Concerning completeness, it is in general not an easy task to establish if a given logic enjoys that property. But in the next section we prove a quite general completeness theorem.

6 Completeness

In this section, we assume as given once and for all a logic $\mathcal{L}$ with underlying Hilbert framework $\mathcal{H}$; what follows applies to every signature Σ within the logic. Moreover, since we are here interested in completeness issues (see Definition 5.16) we will consider only formulae from $L(\Sigma, X)$. This means in particular that provisos will not appear; in fact, for $\Gamma \subseteq L(\Sigma, X)$ and $\gamma \in L(\Sigma, X)$, if, for instance, $\Gamma \vdash_{\mathrm{d}} \gamma : \pi$, then $\Gamma \vdash_{\mathrm{d}} \gamma : \mathbf{1}_{\Sigma}$.

Definition 6.1 Within $H(\Sigma)$, let $\Gamma \subseteq L$ and $\varphi \in L$. We say that Γ is φ-*consistent* iff $\Gamma \not\vdash_{\mathrm{d}}^{\Sigma} \varphi$. The set Γ is said to be a φ-*maximal consistent set* (φ-m.c.s.) if it is φ-consistent and no proper extension of it is φ-consistent. We say that Γ is *consistent* or *maximal consistent* if, respectively, there is a φ such that Γ is φ-consistent, or there is a φ such that Γ is φ-maximal consistent. Any φ-m.c.s. including Γ is said to be a φ-*maximal consistent extension* (φ-m.c.e.) of Γ. Every φ-m.c.e. of Γ is said to be an m.c.e. of Γ. $\triangle$

The proof of the following *Lindenbaum Lemma* is based on the usual construction.[1]

Lemma 6.2 Within $H(\Sigma)$, for every consistent set Γ and every φ such that $\Gamma \not\vdash_{\mathrm{d}} \varphi$, there exists an φ-m.c.e. of Γ. In particular, every consistent set can be extended to an m.c.s.. $\triangle$

Lemma 6.3 Within $H(\Sigma)$, if Γ is an m.c.s., then, for all terms t, t', $t = t' \in \Gamma$ iff $t \neq t' \notin \Gamma$.

Proof: By Clause 1 in Definition 5.14, there exists no consistent set containing both $t = t'$ and $t \neq t'$. Assume that Γ is a φ-m.c.s. such that $t = t' \notin \Gamma$ and $t \neq t' \notin \Gamma$, then $\Gamma \cup \{t = t'\} \vdash_{\mathrm{d}} \varphi$ and $\Gamma \cup \{t \neq t'\} \vdash_{\mathrm{d}} \varphi$, and hence, by Clause 2 in Definition 5.14, $\Gamma \vdash_{\mathrm{d}} \varphi$, which contradicts the assumption. QED

Definition 6.4 A logic $\mathcal{L}$ is said to be *full* iff for every signature $\Sigma \in \mathit{Sig}$ and Σ-structure s for the Hilbert system $H(\Sigma)$ there is a model $m \in M$ in $S(\Sigma)$ such that $A(m) = s$. $\triangle$

Example 6.5 *Modal K first-order logic.* The modal first-order logic presented in Example 5.17 is not full. But, for each $\Sigma(I, F, P)$, we can enrich it with all $\Sigma(I, F, P)$-structures for the Hilbert system $H(\Sigma(I, F, P))$.

Theorem 6.6 *Completeness.* Every full, congruent, persistent, and uniform logic for equality and inequality is complete.

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[1] The consistency of a set Γ is often defined as $\Gamma^{\vdash_{\mathrm{d}}} \neq L$, but, for arbitrary Hilbert calculi, in general it is not true that, if $\Gamma^{\vdash_{\mathrm{d}}} \neq L$, then there exists a maximal extension of Γ with this property. This holds, however, if there exists a formula $\perp$ such that $\{\perp\}^{\vdash_{\mathrm{d}}} = L$ (see [16]). By Clause 1 in Definitions 5.13 and 5.14, in Hilbert calculi for inequality, the role of $\perp$ can be played by $t \neq t$ for any term t. Even if we will be mainly interested in these Hilbert calculi, we prefer using the notion of consistency given in Definition 6.1 for the sake of generality.

This theorem will be proved using the Henkin construction below which is based on the following auxiliary concepts and lemmas. The Hilbert framework $\mathcal{H} = \langle Sig, H\rangle$ is assumed to be congruent, persistent, uniform, and for equality and inequality.

From here on we consider fixed a signature $\Sigma \in Sig$ and H may stand for $H(\Sigma)$. Recall also that we may use L for $L(\Sigma, X)$, cL for the closed formulae in $L(\Sigma, X)$, T for $T(\Sigma, X)$ and gT for the ground terms in $T(\Sigma, X)$.

Given any set E such that $F_0 \cap E = \emptyset$ and $I \cap E = \emptyset$, we denote by Σ_E the signature in Sig obtained from Σ by replacing I by $I \cup E$ and we refer to $H(\Sigma_E)$ simply as H_E. Similarly, we may use L_E for $L(\Sigma_E, X)$, cL_E for the closed formulae in $L(\Sigma_E, X)$, T_E for $T(\Sigma_E, X)$ and gT_E for the ground terms in $T(\Sigma_E, X)$.

Definition 6.7 Given any set Γ of formulae in L_E, the *Q-kernel* and the *O-kernel* of Γ, written $K_Q(\Gamma)$ and $K_O(\Gamma)$, are defined by

$$\begin{aligned}
K_Q(\Gamma) &= \{\varphi \in \Gamma : \varphi \text{ is a first-order formula and } \varphi \in cL\} \cup \\
&\quad \{t = d \in \Gamma : t \in gT \text{ and } d \in E\} \cup \\
&\quad \{d \neq d' : \text{ distinct } d, d' \in E \text{ and} \\
&\qquad\qquad \exists t, t' \in gT (t = d \in \Gamma \text{ and } t' = d' \in \Gamma)\}\,; \\
K_O(\Gamma) &= \{t = d \in \Gamma : t \in X \cup I \text{ and } d \in E\} \cup \\
&\quad \{t \neq t' \in \Gamma : t, t' \in X \cup I\} \cup \\
&\quad \{d \neq d' : \text{ distinct } d, d' \in E \text{ and} \\
&\qquad\qquad \exists t, t' \in X \cup I (t = d \in \Gamma \text{ and } t' = d' \in \Gamma)\}\,.
\end{aligned}$$

Lemma 6.8 Within H_E, for every set Γ of formulae in L_E such that $(K_Q(\Gamma) \cup K_O(\Gamma)) \subseteq \Gamma$,

$$\left(\Gamma \cup (K_Q(\Gamma))^{\vdash_{\mathrm{Qp}}} \cup (K_O(\Gamma))^{\vdash_{\mathrm{Op}}}\right)^{\vdash_{\mathrm{d}}} = \Gamma^{\vdash_{\mathrm{d}}}$$

Proof: Consider the instances of (VP) and (HP) in Definition 5.11 in which $\Gamma^{\vdash_{\mathrm{p}}} = \emptyset^{\vdash_{\mathrm{p}}}$ and Ψ is, respectively, $K_Q(\Gamma)$ and $K_O(\Gamma)$. In these cases, (VP) and (HP) imply $(K_Q(\Gamma))^{\vdash_{\mathrm{Qp}}} = (K_Q(\Gamma))^{\vdash_{\mathrm{d}}}$ and $(K_O(\Gamma))^{\vdash_{\mathrm{Op}}} = (K_O(\Gamma))^{\vdash_{\mathrm{d}}}$, because $\emptyset^{\vdash_{\mathrm{p}}}$ is contained in any set of the form $K^{\vdash_{\mathrm{Qp}}}$ or $K^{\vdash_{\mathrm{Op}}}$ and because the sets $K_Q(\Gamma)$ and $K_O(\Gamma)$ fulfill the clauses for Ψ of that definition. Then, from the assumption $\Gamma \cup (K_Q(\Gamma))^{\vdash_{\mathrm{Qp}}} \cup (K_O(\Gamma))^{\vdash_{\mathrm{Op}}} \vdash_{\mathrm{d}} \varphi$, we have $\Gamma \cup (K_Q(\Gamma))^{\vdash_{\mathrm{d}}} \cup (K_O(\Gamma))^{\vdash_{\mathrm{d}}} \vdash_{\mathrm{d}} \varphi$ which yields $\Gamma \vdash_{\mathrm{d}} \varphi$ because $K_Q(\Gamma) \subseteq \Gamma$ and $K_O(\Gamma) \subseteq \Gamma$. QED

Definition 6.9 Let E be a set such that $F_0 \cap E = \emptyset$ and $I \cap E = \emptyset$. A set $\Gamma \subseteq L_E$ is said to be an *E-Henkin set* iff:

1. Γ is an m.c.s. in H_E;
2. for every term $t \in T$, there is a $d \in E$ such that $t = d$ is in Γ;
3. for every $d \in E$, there is a term $t \in T$ such that $t = d \in \Gamma$;
4. $\{d \neq d' : d, d' \in E\} \subseteq \Gamma$.

The set Γ is said to be an *E-pre-Henkin set* iff Clauses 3 and 4 above are fulfilled. $\triangle$

The elements of E in an E-Henkin set will be called *witnesses*. If usual first-order logic is considered, the definition of Henkin set given above is different from the traditional one in the sense that, if Γ is an E-Henkin set and $\exists x\varphi(x) \in \Gamma$, then the definition does not guarantee that there is a witness d such that $\varphi(d)$ belongs to Γ. This seeming departure from the usual construction will be explained in Remark 6.12 below.

Given any set $\Gamma \subseteq L_E$ and any $\delta \in E^{X\cup I}$ we write $\delta \subseteq \Gamma$ iff, for every $t \in X \cup I$, $t = \delta(t)$ is in Γ. If Γ is a E-Henkin set, there is a unique $\delta \in E^{X\cup I}$ such that $\delta \subseteq \Gamma$ which we denote by δ_Γ. Moreover, it can be easily verified that, if Γ is both an E_1-Henkin set and an E_2-Henkin set, then $E_1 = E_2$.

Lemma 6.10 *Henkin extension.* Assume that: 1) D is a set with cardinality greater than T such that $F_0 \cap D = \emptyset$ and $I \cap D = \emptyset$, 2) E is a possibly empty subset of D, 3) Γ is a E-pre-Henkin set, and 4) Γ is φ-consistent in H_E. Then, there exist a set E^* and a set $\Gamma^* \subseteq L_{E^*}$ such that: i) $E \subseteq E^* \subseteq D$, ii) Γ^* is φ-m.c.e. of Γ in H_{E^*}, and iii) Γ^* is an E^*-Henkin set.

Proof: Since Γ fulfills Clause 3 of Definition 6.9, also the cardinality of $D \setminus E$ is greater than that of T and hence we can consider an infinite sequence $d_0, d_1, \ldots$ of elements of $D \setminus E$. Given any enumeration $t_0, t_1, \ldots$ of T, we define the sequences $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$ and $E_0 \subseteq E_1 \subseteq \ldots$ of subsets of L_D and of D, respectively, such that, for each k,

$$(*) \quad \Gamma_k \text{ is a } \varphi\text{-m.c.e. of } \Gamma \text{ within } H_{E_k}\text{, and}$$
$$(**) \quad \Gamma_k \text{ is a } E_k\text{-pre-Henkin set.}$$

We let E_0 be E and Γ_0 be a φ-m.c.e. of Γ within H_E (which exists by Lemma 6.2); thus, $(*)$ and $(**)$ hold for $k = 0$. Assuming inductively that $(*)$ and $(**)$ hold for an arbitrary k, the sets Γ_{k+1} and E_{k+1} are defined according to the following cases.

Case 1: for some $d \in E_k$, $d = t_k \in \Gamma_k$. We set $\Gamma_{k+1} = \Gamma_k$ and $E_{k+1} = E_k$.

Case 2: for no $d \in E_k$, $d = t_k \in \Gamma_k$. We set $E_{k+1} = E_k \cup \{d_k\}$ and we let Γ_{k+1} be any φ-m.c.e. of $\Gamma_k^+ = \Gamma_k \cup \{t_k = d_k\} \cup \{d \neq d_k : d \in E_k\}$ within $H_{E_{k+1}}$.

By Lemma 6.2, Γ_{k+1} exists if the set Γ_k^+ is φ-consistent in $H_{E_{k+1}}$. Since Γ_k is a φ-m.c.s. in H_{E_k} and $t_k = d \notin \Gamma_k$ for all $d \in E_k$, Lemma 6.3 implies $t_k \neq d \in \Gamma_k$ for all $d \in E_k$, and hence Γ_k^+ is φ-consistent in $H_{E_{k+1}}$ if and only if such is $\Gamma_k \cup \{t_k = d_k\}$ (because each inequality $d \neq d_k$ can be inferred from $t_k \neq d$ and $t_k = d_k$). If $\Gamma_k \cup \{t_k = d_k\} \vdash_{\mathrm{d}} \varphi$ in $H_{E_{k+1}}$, Clause 5 in Definition 5.13 implies $\Gamma_k \vdash_{\mathrm{d}} \varphi$ in $H_{E_{k+1}}$, because d_k does not occur in Γ_k and in φ. By (the proof of) Proposition 5.10, we can replace every occurrence of d_k in the derivation of φ from Γ_k in $H_{E_{k+1}}$ by a variable x which does not occur in that derivation; in this way, a derivation of φ from Γ_k in H_{E_k} is obtained. This contradicts the inductive assumption that Γ_k is φ-consistent in H_{E_k}. It is straightforward to verify that Γ_{k+1} has the properties $(*)$ and $(**)$.

Set $E^* = \cup_{k\in\mathbb{N}} E_k$. The envisaged E^*-Henkin set is given by $\Gamma^* = \cup_{k\in\mathbb{N}}\Gamma_k$. Indeed, by construction, Γ^* contains Γ and fulfills Clauses 2 to 4 in Definition 6.9. As for Clause 1 in that definition, observe that the set Γ^* is φ-consistent within H_{E^*} because such is any Γ_k and any derivation in H_{E^*} is also a derivation in some H_{E_k}. Furthermore, given any formula $\psi \notin \Gamma^*$, we can consider a k such that $\psi \in L_{E_k}$. Since $\psi \notin \Gamma_k$ and Γ_k is an φ-m.c.s. within H_{E_k}, we have $\Gamma_k \cup \{\psi\} \vdash_{\mathrm{d}} \varphi$ in H_{E_k}, which implies $\Gamma^* \cup \{\psi\} \vdash_{\mathrm{d}} \varphi$ in H_{E^*}. QED

The Henkin construction

Given a consistent and p-deductively closed set $\Gamma_0 \subseteq L$ and a set D with cardinality greater than that of T, such that $F_0 \cap D = \emptyset$ and $I \cap D = \emptyset$, we define an appropriate structure

$$s = \langle U, \Delta, W, \alpha, \omega, D, \mathcal{E}, \mathcal{B}, [\cdot]\rangle$$

for the Hilbert calculus H at hand as follows. We set

$$U \ = \ \{u \subseteq L_D : u \text{ is a } E\text{-Henkin set for some } E \subseteq D \text{ and } \Gamma_0 \subseteq u\}.$$

If $u \in U$ is an E-Henkin set, the set E and the Hilbert calculus H_E will be referred to as E_u and H_u, respectively.

Lemma 6.11 For every $u \in U$, we have

1. $K_Q(u) \subseteq u$ and $K_O(u) \subseteq u$;
2. $((K_Q(u))^{\vdash_{\mathrm{Qp}}} \cup \Gamma_0)^{\vdash_{\mathrm{d}}} \subseteq u$ and $((K_O(u))^{\vdash_{\mathrm{Op}}} \cup \Gamma_0)^{\vdash_{\mathrm{d}}} \subseteq u$ within H_u;
3. the sets $K_Q(u)$, $K_O(u)$, and $K_Q(u) \cup K_O(u)$ are E-pre-Henkin sets, where E is the smallest subset of D such that all its elements occur in, respectively, $K_Q(u)$, or $K_O(u)$, or $K_Q(u) \cup K_O(u)$;
4. there are infinitely many elements of D which do not occur in u.

Proof: 1. By the definition of Henkin set. 2. By Lemma 6.8, taking into account that u contains Γ_0 and is d-deductively closed. 3. By Definition 6.7 and the definition of pre-Henkin set. 4. By Clause 3 in Definition 6.9, we have that the cardinality of the set of elements of D which occur in u is smaller than the (infinite) cardinality of D. QED

The sets W and Δ, and the functions ω and α are defined by

$$W \ = \{K_Q(u) : u \in U\} \qquad \Delta \ = \{\delta_u : u \in U\} \tag{6.1}$$

$$\omega(u) = K_Q(u) \qquad \alpha(u) = \delta_u \tag{6.2}$$

Remark 6.12 *Witnesses in Henkin constructions.* As observed above, if the logic at hand is usual first-order logic, then Definition 6.9 does not guarantee that, if $\exists x\varphi(x) \in u \in U$, then $\varphi(d) \in u$ for some d in D. This can be explained by observing that, in our construction, the role of the usual Henkin sets is essentially played by the sets U_w. It is true, in fact, that if $\exists x\varphi(x) \in u \in U_w$, then there is a $u' \in U_w$ and a $d \in D$ such that $\varphi(d) \in u'$. In order to prove this, observe first that the set $\Gamma_0 \cup K_Q(u) \cup \{\varphi(x)\}$ is consistent because otherwise $\Gamma_0 \cup K_Q(u) \vdash_{\mathrm{d}} \neg\varphi(x)$ and, since Γ_0 is p-deductively closed and $K_Q(u)$ is composed of closed formulae, $\Gamma_0 \cup K_Q(u) \vdash_{\mathrm{d}} \forall x \neg\varphi(x)$, which is impossible because $\Gamma_0 \cup K_Q(u) \cup \{\exists x\varphi(x)\} \subseteq u$ and u is consistent. Then, we can consider a Henkin extension u' of $\Gamma_0 \cup K_Q(u) \cup \{\varphi(x)\}$ which is in U_w and, for some $d \in D$, contains $x = d$. From the properties of equality, we also have $\varphi(d) \in u'$. $\triangle$

Observe that there is a one-to-one correspondence between the set Δ and the set of all $K_O(u)$ such that $u \in U$; in fact, all the equalities $t = \delta_u(t)$ belong to $K_O(u)$ and, given δ_u, the set $K_O(u)$ is the set of all equalities and inequalities which can be derived from the set $\{t = \delta_u(t) : t \in X \cup I\}$. Thus, the sets U_w and U_δ considered in Definition 3.1 fulfill the following equalities, in which u is any element of U such that, respectively, $\omega(u) = w$ and $\delta(u) = \delta$.

$$U_w = \{u' : K_Q(u') = K_Q(u)\} \qquad U_\delta = \{u' : K_O(u') = K_O(u)\} \tag{6.3}$$

On the basis of (6.3), the set $U_{w\delta}$ turns out to fulfill

$$U_{w\delta} = \{u : K_Q(u) = w \text{ and } \delta_u = \delta\} \tag{6.4}$$

The sets E_w, E_δ, and $E_{w\delta}$ are respectively defined as $\cap\{E_u : u \in U_w\}$, $\cap\{E_u : u \in U_\delta\}$, and $\cap\{E_u : u \in U_{w\delta}\}$, so that we have that $E_w \cup E_\delta \subseteq E_{w\delta}$. The calculi H_{E_w}, H_{E_δ}, $H_{E_{w\delta}}$ will be also referred to as H_w, H_δ, and $H_{w\delta}$.

The extension $|\gamma|$ of a formula γ in L and the extension $|t|$ of term t in T, are functions from U into, respectively, $\{0,1\}$ and D:

$$\begin{aligned} |\gamma|(u) = 1 &\text{ iff } \gamma \in u \\ |t|(u) = d &\text{ iff } t = d \in u \end{aligned} \tag{6.5}$$

It is worth observing that, if u and u' belong to the same U_w, then $w = K_Q(u) = K_Q(u')$ and hence, for every closed first-order formula γ of L and every ground term t in T, $|\gamma|(u) = |\gamma|(u')$ and $|t|(u) = |t|(u')$. As far as extensions of formulae are concerned, we will shift freely from the functional notation to the set notation, that is, we will often write $u \in |\gamma|$ instead of $|\gamma|(u) = 1$. The sets $\mathcal{B}$ and $\mathcal{E}$ are defined by:

$$\mathcal{B} \;=\; \{|\gamma| : \gamma \in L\} \text{ and } \mathcal{E} \;=\; \{|t| : t \in T\} \tag{6.6}$$

Given any formula $\varphi \in \Gamma_0$, since any element of U contains Γ_0, we have that $|\varphi| = U$ and hence U is an element of $\mathcal{B}$. On the basis of (6.6), we will use, possibly indexed, $|\gamma|$ and $|t|$ to denote elements of $\mathcal{B}$ and of $\mathcal{E}$, respectively. Thus, setting

$$\begin{aligned} |\gamma|_w &= |\gamma| \cap U_w \; (= \{u : K_Q(u) = w \text{ and } \gamma \in u\}) \text{ and} \\ |\gamma|_\delta &= |\gamma| \cap U_\delta \; (= \{u \in U : \gamma \in u \text{ and } \delta_u = \delta\}) \end{aligned} \tag{6.7}$$

the sets $\mathcal{B}_w$ and $\mathcal{B}_\delta$ can be written as

$$\mathcal{B}_w = \{|\gamma|_w : \gamma \in L\} \quad \text{and} \quad \mathcal{B}_\delta = \{|\gamma|_\delta : \gamma \in L\} \tag{6.8}$$

Finally, by (6.7) and (6.8), we have

$$|\gamma|_{w\delta} = |\gamma|_w \cap |\gamma|_\delta \quad \text{and} \quad \mathcal{B}_{w\delta} = \{|\gamma|_{w\delta} : \gamma \in L\}. \tag{6.9}$$

The construction of the envisaged Σ-structure is accomplished by the following definition of the interpretation map $[\cdot]$. We first define the interpretation of the elements of X, I, F_k, and of P_k:

- for $x \in X$, $i \in I$, and $\delta \in \Delta$,

$$[x]_\delta = \delta(x)\,, \qquad [i]_\delta = \delta(i)\,; \tag{6.10}$$

- for $f \in F_k$ $(k \geq 0)$ and $u \in U_w$,

$$[f]_w(|t_1|(u), \ldots, |t_k|(u)) = |f(t_1, \ldots, t_k)|(u)\,^2; \tag{6.11}$$

- for every $u \in U$,

$$[=](|t_1|(u), |t_2|(u)) = 1 \text{ iff } |t_1|(u) = |t_2|(u)\,; \tag{6.12}$$

$$[\neq](|t_1|(u), |t_2|(u)) = 1 \text{ iff } |t_1|(u) \neq |t_2|(u)\,; \tag{6.13}$$

- for $p \in P_k$ $(k \geq 0)$ and $u \in U_w$,

$$[p]_w(|t_1|(u), \ldots, |t_k|(u)) = |p(t_1, \ldots, t_k)|(u)\,. \tag{6.14}$$

As far as (6.10) is concerned, we have to show that, according to Definition 3.1, for every $i \in I$, every w, and $u, u' \in U_w$, $[i]_{\alpha(u)} = [i]_{\alpha(u')}$. This follows immediately by observing that $u, u' \in U_w$ implies $K_Q(u) = K_Q(u')$ and that these sets contain all the equalities of the form $i = \delta_u(i)$ and $i = \delta_{u'}(i)$, which implies $\delta_u(i) = \delta_{u'}(i)$.

In order to show that $[f]_w$, $[p]_w$ are well defined, we use some properties of equality given in Definition 5.13. Observe first that, if $|t_i|(u) = |t'_i|(u)$, then there is a $d \in D$ such that $t_i = d \in u$ and $t'_i = d \in u$, which implies in turn $t_i = t'_i \in u$. So, for some $d \in D$, $f(t_1, \ldots, t_i, \ldots, t_k) = d \in u$ iff $f(t_1, \ldots, t'_i, \ldots, t_k) = d \in u$, and $p(t_1, \ldots, t_i, \ldots, t_k) \in u$ iff $p(t_1, \ldots, t'_i, \ldots, t_k) \in u$. Moreover, assume

[2] More precisely, $[f]_w$ is defined by (6.11) for the relevant tuples and chosen arbitrarily elsewhere. Observe that the value of the interpretation elsewhere is irrelevant in Definition 3.2, and, therefore, irrelevant to both semantic entailments. The same applies to the interpretation of the other symbols.

that the tuples $\langle |t_1|(u), \ldots, |t_k|(u)\rangle$ and $\langle |t_1|(u'), \ldots, |t_k|(u')\rangle$ coincide for $u, u' \in U_w$ and let them be $\langle d_1, \ldots, d_k\rangle$. This means that, for i=1 to k, t_i=$d_i \in u$ and t_i=$d_i \in u'$. This implies that, for every d, $f(t_1, \ldots, t_k) = d \in u$ iff $f(t_1, \ldots, t_k) = d \in u'$ and that $p(t_1, \ldots, t_k) \in u$ iff $p(t_1, \ldots, t_k) \in u'$.

The functions $[c]_{w\delta}$, $[qx]_w$, and $[o]_\delta$ are defined by

- for every $c \in C_k$ $(k \geq 0)$, $w \in W$, and any assignment δ,

$$[c]_{w\delta}(|\gamma_1|_{w\delta}, \ldots, |\gamma_k|_{w\delta}) \;=\; |c(\gamma_1, \ldots, \gamma_k)|_{w\delta}\,; \tag{6.15}$$

- for every $q \in Q_k$, $x \in X$, and $w \in W$,

$$[qx]_w(|\gamma_1|_w, \ldots, |\gamma_k|_w) \;=\; |qx(\gamma_1, \ldots, \gamma_k)|_w\,; \tag{6.16}$$

- for every $o \in O_k$ and any assignment δ,

$$[o]_\delta(|\gamma_1|_\delta, \ldots, |\gamma_k|_\delta) \;=\; |o(\gamma_1, \ldots, \gamma_k)|_\delta\,. \tag{6.17}$$

The proof that $[c]_{w\delta}$ is well defined passes through the proof that, for all formulae γ and γ',

$$|\gamma|_{w\delta} = |\gamma'|_{w\delta} \text{ iff } \left\{ \begin{array}{l} \text{for all } u \in U_{w\delta},\ \Gamma_0 \cup K_Q(u) \cup K_O(u), \gamma \vdash_{\mathrm{d}} \gamma' \\ \text{for all } u \in U_{w\delta},\ \Gamma_0 \cup K_Q(u) \cup K_O(u), \gamma' \vdash_{\mathrm{d}} \gamma \end{array} \right. \text{ in } H_{w\delta} \tag{6.18}$$

If $|\gamma|_{w\delta} \neq |\gamma'|_{w\delta}$, then there is a $u' \in U_{w\delta}$ such that, e.g., $\gamma \in u'$ and $\gamma' \notin u'$. Since u' is d-deductively closed, we have $u' \not\vdash_{\mathrm{d}} \gamma'$ in $H_{u'}$, which implies $\Gamma_0 \cup K_Q(u) \cup K_O(u), \gamma \not\vdash_{\mathrm{d}} \gamma'$ in $H_{w\delta}$ because $\Gamma_0 \cup K_Q(u') \cup K_O(u') \cup \{\gamma\} \subseteq u'$, $K_Q(u) = K_Q(u')$, $K_O(u) = K_O(u')$, and $E_{w\delta} \subseteq E_{u'}$.

Conversely, assume, e.g., $\Gamma_0 \cup K_Q(u) \cup K_O(u), \gamma \not\vdash_{\mathrm{d}} \gamma'$ in $H_{w\delta}$. The set $\Gamma_0 \cup K_Q(u) \cup K_O(u) \cup \{\gamma\}$ is a E-pre-Henkin set, where E is the set of all elements of D which occur in $K_Q(u) \cup K_O(u)$, because no element of D occurs in Γ_0 and in γ. By Lemma 6.10, we can consider a Henkin set u' such that $\Gamma_0 \cup K_Q(u) \cup K_O(u) \subseteq u'$, $\gamma \in u'$ and $\gamma' \notin u'$. The set u' belongs to $U_{w\delta}$ and hence $|\gamma|_{w\delta} \neq |\gamma'|_{w\delta}$. This concludes the proof of (6.18).

Assume now $|\gamma_i|_{w\delta} = |\gamma_i'|_{w\delta}$ for $i = 1$ to k. For any $u \in U_{w\delta}$, (6.18) implies that, in $H_{w\delta}$,

$$\left\{ \begin{array}{l} \Gamma_0 \cup K_Q(u) \cup K_O(u), \gamma_i \vdash_{\mathrm{d}} \gamma_i' \\ \Gamma_0 \cup K_Q(u) \cup K_O(u), \gamma_i' \vdash_{\mathrm{d}} \gamma_i \end{array} \right.$$

Then, *a fortiori*

$$\left\{ \begin{array}{l} (\Gamma_0 \cup K_Q(u))^{\vdash_{\mathrm{Qp}}} \cup (\Gamma_0 \cup K_O(u))^{\vdash_{\mathrm{Op}}}, \gamma_i \vdash_{\mathrm{d}} \gamma_i' \\ (\Gamma_0 \cup K_Q(u))^{\vdash_{\mathrm{Qp}}} \cup (\Gamma_0 \cup K_O(u))^{\vdash_{\mathrm{Op}}}, \gamma_i' \vdash_{\mathrm{d}} \gamma_i \end{array} \right.$$

and, by Clause 1 in Definition 5.12,

$$\left\{ \begin{array}{l} (\Gamma_0 \cup K_Q(u))^{\vdash_{\mathrm{Qp}}} \cup (\Gamma_0 \cup K_O(u))^{\vdash_{\mathrm{Op}}}, c(\gamma_1, \ldots, \gamma_k) \vdash_{\mathrm{d}} c(\gamma_1', \ldots, \gamma_k') \\ (\Gamma_0 \cup K_Q(u))^{\vdash_{\mathrm{Qp}}} \cup (\Gamma_0 \cup K_O(u))^{\vdash_{\mathrm{Op}}}, c(\gamma_1', \ldots, \gamma_k') \vdash_{\mathrm{d}} c(\gamma_1, \ldots, \gamma_k) \end{array} \right.$$

Observe now that $K_Q(u)$ is a set of first-order closed formulae in L_E and hence it fulfills the requirement for Ψ in (VP) in Definition 5.11 and that, since the elements of E are rigid designators in $H_{w\delta}$, the set $K_O(u)$ fulfills the requirement for Ψ in (HP) in that definition. Moreover, the set Γ_0 is p-deductively closed. Thus,

$$\begin{cases} \Gamma_0 \cup K_Q(u) \cup K_O(u), c(\gamma_1, \ldots, \gamma_k) \vdash_{\mathrm{d}} c(\gamma_1', \ldots, \gamma_k') \\ \Gamma_0 \cup K_Q(u) \cup K_O(u), c(\gamma_1', \ldots, \gamma_k') \vdash_{\mathrm{d}} c(\gamma_1, \ldots, \gamma_k) \end{cases}$$

and hence (6.18) implies $|c(\gamma_1, \ldots, \gamma_k)|_{w\delta} = |c(\gamma_1', \ldots, \gamma_k')|_{w\delta}$.

In order to prove that the functions $[qx]_w$ and $[o]_\delta$ are well defined, we proceed in a way quite similar to that we used above for $[c]_{w\delta}$. We briefly describe the main steps of the proof only for $[qx]_w$. We first prove that, for all $\gamma, \gamma' \in L$, every $w \in W$, and every $u \in U_w$,

$$|\gamma|_w = |\gamma'|_w \ \text{ iff } \begin{cases} \Gamma_0 \cup K_Q(u), \gamma \vdash_{\mathrm{d}} \gamma' \\ \Gamma_0 \cup K_Q(u), \gamma' \vdash_{\mathrm{d}} \gamma \end{cases} \text{ in } H_w$$

and, assuming $|\gamma_i|_w = |\gamma_i'|_w$, for i=1 to k, we prove that, in H_w,

$$\begin{cases} (\Gamma_0 \cup K_Q(u))^{\vdash_{\mathrm{Qp}}}, qx\,(\gamma_1, \ldots, \gamma_k) \vdash_{\mathrm{d}} qx\,(\gamma_1', \ldots, \gamma_k') \\ (\Gamma_0 \cup K_Q(u))^{\vdash_{\mathrm{Qp}}}, qx\,(\gamma_1', \ldots, \gamma_k') \vdash_{\mathrm{d}} qx\,(\gamma_1, \ldots, \gamma_k) \end{cases}$$

The proof of these derivability results is quite similar to that of the corresponding results for $[c]_{w\delta}$. Then, we observe that $K_Q(u)$ is a set of closed formulae in L_E and hence, by (VP) in Definition 5.11, the following deductions hold in H_w,

$$\begin{cases} \Gamma_0 \cup K_Q(u), qx\,(\gamma_1, \ldots, \gamma_k) \vdash_{\mathrm{d}} qx\,(\gamma_1', \ldots, \gamma_k') \\ \Gamma_0 \cup K_Q(u), qx\,(\gamma_1', \ldots, \gamma_k') \vdash_{\mathrm{d}} qx\,(\gamma_1, \ldots, \gamma_k) \end{cases}$$

which implies $|qx(\gamma_1, \ldots, \gamma_k)|_w = |qx(\gamma_1', \ldots, \gamma_k')|_w$.

End of Henkin construction.

In the next lemmas, we will consider the evaluations $[\![t]\!]^s_\tau$ and $[\![\varphi]\!]^s_\phi$, where s is the structure defined in the Henkin construction. Since there will be no ambiguity, in the sequel we will often simply write $[\![t]\!]$ and $[\![\varphi]\!]$ for these evaluations.

Lemma 6.13 For every term $t \in T$, $|t| = [\![t]\!]$.

Proof: Consider any term $t \in X \cup I$ and assume $[\![t]\!](u) = d$. This means that $[t]_\delta = d$, i.e., $\delta(t) = d$, where δ is $\alpha(u)$, that is, δ is δ_u, the only assignment contained in u. This implies $t = d \in u$ and, by (6.5), $|t|(u) = d$.

Let f be any element of F_0. By Definition 3.2, for every $u \in U$, $[\![f]\!](u) = [f]_{\omega(u)}$, and, by (6.11), $[f]_{\omega(u)} = |f|(u)$.

Let t be $f(t_1, \ldots, t_k)$ and assume, as inductive hypothesis, that the claim holds for $t_1, \ldots, t_k$. By Definition 3.2, for every $u \in U$, $[\![f(t_1, \ldots, t_k)]\!](u) = [f]_{\omega(u)}([\![t_1]\!](u), \ldots, [\![t_k]\!](u))$. Then, by the inductive hypothesis, $[\![f(t_1, \ldots, t_k)]\!](u) = [f]_{\omega(u)}(|t_1|(u), \ldots, |t_k|(u))$ and hence the claim follows by (6.11). QED

Theorem 6.14 For every formula $\varphi \in L$, $|\varphi| = [\![\varphi]\!]$.

Proof: If φ is an element, p, of P_0, then, for every u, $[\![p]\!](u) = [p]_{\omega(u)} = |p|(u)$. If φ is $p(t_1, \ldots, t_k)$, then, for every u, $[\![p(t_1, \ldots, t_k)]\!](u) = \widehat{p}([\![t_1]\!], \ldots, [\![t_k]\!])(u)$ which is equal to $[p]_{\omega(u)}(|t_1|(u), \ldots, |t_k|(u))$ by Lemma 6.13 and Definition 3.2. Thus, (6.14) implies $[\![p(t_1, \ldots, t_k)]\!](u) = |p(t_1, \ldots, t_k)|(u)$.

Assume now inductively that, for $i = 1$ to k, $[\![\varphi_i]\!] = |\varphi_i|$. In the rest of the proof, we will use Definition 3.2 and Lemma 6.13, as well as (6.15), (6.16), and (6.17), without referring explicitly to them.

Case 1: φ is $c(\varphi_1, \ldots, \varphi_k)$, where $c \in C_k$. So, for $u \in U_{w\delta}$, $[\![c(\varphi_1, \ldots, \varphi_k)]\!](u) = [c]_{w\delta}(|\varphi_1|(u), \ldots, |\varphi_k|(u)) = |c(\varphi_1, \ldots, \varphi_k)|(u)$.

Case 2: φ is $qx(\varphi_1, \ldots, \varphi_k)$, where $q \in Q_k$. So, for $u \in U_w$, $u \in [\![qx(\varphi_1, \ldots, \varphi_k)]\!]$ iff $u \in [qx]_w(|\varphi_1| \cap U_w, \ldots |\varphi_k| \cap U_w)$ iff $u \in [qx]_w(|\varphi_1|_w, \ldots |\varphi_k|_w) = |qx(\varphi_1, \ldots, \varphi_k)|_w$. Then, we can conclude that $u \in [\![\varphi]\!]$ iff $u \in |\varphi| \cap U_w$. Since $u \in U_w$, $u \in [\![\varphi]\!]$ is equivalent to $u \in [\![\varphi]\!] \cap U_w$ and the equality $[\![\varphi]\!] = |\varphi|$ follows immediately by observing that the sets U_w are pairwise disjoint.

Case 3: φ is $o(\varphi_1, \ldots, \varphi_k)$, where $o \in O_k$. The proof is quite similar to that of the previous case. QED

Corollary 6.15 The structure s defined in the Henkin construction is appropriate for H.

Proof: Let $\langle \{\varphi_1, \ldots, \varphi_k\}, \varphi \rangle$ be an instance of a rule in R_{d} and assume that, for some u in U and for i=1 to k, $u \in [\![\varphi_i]\!]$, which implies $u \in |\varphi_i|$, by Lemma 6.14. Then, for each i, $\varphi_i \in u$, and hence $\varphi \in u$ because this set is d-deductively closed. Thus, we can also conclude $u \in [\![\varphi]\!]$.

As far as rules in R_{p} are concerned, observe first that, for every $\gamma \in L$,

$$|\gamma| = U \quad \text{iff} \quad \gamma \in \Gamma_0 \tag{$*$}$$

In fact, if $\gamma \in \Gamma_0$, then $\gamma \in u$ for every $u \in U$ and hence $|\gamma| = U$ by (6.5). If, conversely, $\gamma \notin \Gamma_0$, then $\Gamma_0 \not\vdash_{\mathrm{p}} \gamma$ because Γ_0 is p-deductively closed. This implies $\Gamma_0 \not\vdash_{\mathrm{d}} \gamma$ and hence Γ_0 is γ-consistent; moreover, Γ_0 is a $\emptyset$-pre-Henkin set. Thus, by Lemma 6.10, we can consider an element u of U such that $\gamma \notin u$. This concludes the proof of $(*)$.

Assume that $\langle \{\varphi_1, \ldots, \varphi_k\}, \varphi \rangle$ is an instance of a rule in R_{p} such that, in the structure s, $[\![\varphi_i]\!] = U$, for i=1 to k. By Lemma 6.14, we have $|\varphi_i| = U$, which implies $\varphi_i \in \Gamma_0$ by $(*)$. Since Γ_0 is p-deductively closed, we can conclude $\varphi \in \Gamma_0$, which implies $[\![\varphi]\!] = U$. QED

Proof of the Completeness Theorem: Assume $\Gamma \not\vdash_{\mathrm{d}} \varphi$. Consider a Henkin construction in which $\Gamma_0 = \emptyset^{\vdash_{\mathrm{p}}}$. Since Γ is φ-consistent, we can use Lemma 6.10 with $E = \emptyset$ to conclude that there is a $u \in U$ such that $\Gamma \subseteq u$ and $\varphi \notin u$. Then, by Theorem 6.14, in the structure s, we have $u \in [\![\gamma]\!]$ for all $\gamma \in \Gamma$ and $u \notin [\![\varphi]\!]$.

Assume now $\Gamma \not\vdash_{\mathrm{p}} \varphi$. Consider an arbitrary set D and the structure s defined in the Henkin construction, where Γ_0 is $\Gamma^{\vdash_{\mathrm{p}}}$. The set Γ_0 is φ-consistent

because $\Gamma \not\vdash_{\mathrm{p}} \varphi$ implies $\Gamma^{\vdash_{\mathrm{p}}} \not\vdash_{\mathrm{d}} \varphi$; moreover, Γ_0 is a $\emptyset$-pre-Henkin set. Thus, by Lemma 6.10, there is a $u \in U$ such that $\varphi \notin u$. Since every element of U contains Γ_0, Theorem 6.14 implies $[\![\gamma]\!] = U$ for all $\gamma \in \Gamma$, but $[\![\varphi]\!] \subseteq U \setminus \{u\}$.
QED

Corollary 6.16 *First-order and Modal Completeness.*
1) Let Γ be any set of closed first-order formulae, and let φ be any formula. If, for every structure s and every $w \in W$ in s, $U_w \subseteq [\![\Gamma]\!]$ implies $U_w \subseteq [\![\varphi]\!]$, then $\Gamma \vdash_{\mathrm{Qp}} \varphi$.
2) Let Γ be any set of equalities and inequalities involving only rigid terms, and let φ be any formula. If, for every structure s and every $\delta \in \Delta$ in s, $U_\delta \subseteq [\![\Gamma]\!]$ implies $U_\delta \subseteq [\![\varphi]\!]$, then $\Gamma \vdash_{\mathrm{Op}} \varphi$.

Proof: Assume $\Gamma \not\vdash_{\mathrm{Qp}} \varphi$, so that $\Gamma \not\vdash_{\mathrm{d}} \varphi$. Consider the structure s defined by means of the Henkin construction in which Γ_0 is $\emptyset^{\vdash_{\mathrm{p}}}$. Since Γ is φ-consistent and is trivially a $\emptyset$-pre-Henkin set, we can consider a $u \in U$ such that $\Gamma \subseteq u$ and $\varphi \notin u$. Observe now that $\Gamma \subseteq K_Q(u)$; then, by (6.3), $\Gamma \subseteq u'$ for every $u' \in U_{\omega(u)}$, which implies $U_{\omega(u)} \subseteq [\![\Gamma]\!]$ by Lemma 6.14. The same lemma implies $U_{\omega(u)} \not\subseteq [\![\varphi]\!]$. This concludes the proof of the first claim. The proof of the second one is quite similar. QED

The conditions on Γ in the previous corollary obviously depend on the way in which the sets W and Δ have been defined in the Henkin construction. If this construction had been carried out according to a different characterization of the sets W and Δ, possibly with a finer granularity, then a different, accordingly stronger, version of the previous corollary could have been proved.

7 Fibring

The rest of the paper is dedicated to the problem of fibring first-order based logics. In this section we define precisely what is meant by fibring fob logics, we provide two interesting examples, and we conclude with the result that fibring is conservative with respect to derivation. As we shall see, fibring trivially preserves soundness. We leave until Section 8 the proof that fibring also preserves completeness under some reasonable assumptions.

Before defining the fibring of two fob logics, we need the concept of reduct of a structure under an inclusion of signatures.

Definition 7.1 Given fob signatures $\Sigma \subseteq \Sigma'$ and a Σ'-structure s', the *reduct* of s' to Σ is the Σ-structure $s'|_\Sigma = \langle U', \Delta', W', \alpha', \omega', D', \mathcal{E}', \mathcal{B}', [\cdot]'|_\Sigma \rangle$. $\triangle$

The reduct is a Σ-structure coinciding in all components with the original Σ'-structure barring of course the denotations of the extra symbols which are forgotten. Reducts are essential for relating the models in a fibring with the models in the given logics. Abstracting from the definition in [17], the basic ideas for fibring two logics $\mathcal{L}'$ and $\mathcal{L}''$ can be summarized as follows:

- At the signature level, we should have the symbols from both logics. That is, signatures of the fibring should be unions of signatures of the two given logics.
- For each signature $\Sigma' \cup \Sigma''$ in the fibring, the models should provide denotations for the symbols in that signature and their reducts to Σ' and Σ'' should correspond to models of $\mathcal{L}'$ and $\mathcal{L}''$ for Σ' and Σ'', respectively.
- For each signature $\Sigma' \cup \Sigma''$ in the fibring, the sets of rules should be the unions of the corresponding sets of rules of $\mathcal{L}'$ and $\mathcal{L}''$ for Σ' and Σ'', respectively.

This abstraction is obvious at the signature and deductive system levels. But it is worthwhile to question the abstraction at the semantic level. Adapting from [17], we choose as models of the fibring at $\Sigma' \cup \Sigma''$ all the $\Sigma' \cup \Sigma''$-structures such that their reducts to Σ' and Σ'' correspond to models in $\mathcal{L}'$ and $\mathcal{L}''$, respectively. But, as shown in that paper, this semantics of fibring makes sense (when compared with the original, more intuitive definition in [14]) only in the case of logics endowed with a semantics closed under isomorphic copies and unions of models. The question is: can we safely assume that we are working with such logics? Indeed yes if we decide to work with full logics (Definition 6.4). Such full logics enjoy all the closure properties needed to relate the original definition of fibring with the proposed abstraction. Otherwise, if the given logics are not full one should make them full by endowing them with all structures appropriate for the inference system. This enrichment does not change the logics because the two entailments are unchanged. Therefore, we are justified in adopting the abstraction ideas above to introduce fibring of fob logics as follows:

Definition 7.2 Given two fob logics $\mathcal{L}' = \langle Sig', S', H' \rangle$ and $\mathcal{L}'' = \langle Sig'', S'', H'' \rangle$, their *fibring* is the logic $\mathcal{L}' \cup \mathcal{L}'' = \langle Sig, S, H \rangle$ where:

- $Sig = \{\Sigma' \cup \Sigma'' : \Sigma' \in Sig', \Sigma'' \in Sig''\}$;
- $S(\Sigma' \cup \Sigma'') = \langle M, A \rangle$ where:
  - M is the class of all $\Sigma' \cup \Sigma''$-structures s such that:
    - $s|_{\Sigma'} \in A'(M')$ and $s|_{\Sigma''} \in A''(M'')$;
    - s is appropriate for $H(\Sigma' \cup \Sigma'')$;
  - $A(s) = s$ for each $s \in M$;
- $H(\Sigma' \cup \Sigma'') = H'(\Sigma') \cup H''(\Sigma'')$. △

In each signature $\Sigma' \cup \Sigma''$ of the fibring, the symbols in $\Sigma' \cap \Sigma''$ are said to be *shared*. If no symbols are shared we say that the fibring is *unconstrained* or *free* at that signature. Otherwise, we say that it is *constrained* at that signature by sharing symbols.

In the above definition of the class M in $S(\Sigma' \cup \Sigma'')$, each structure s is required to be appropriate for $H(\Sigma' \cup \Sigma'')$; this is a necessary requirement for having that *appropriateness, and hence soundness, are preserved by fibring.* It may happen, in fact, that $s|_{\Sigma'}$ is appropriate for a rule r' in $H(\Sigma')$, but s is not appropriate for r' in $H(\Sigma' \cup \Sigma'')$: in the richer language there can be new instances of r'. An example of this situation is the FOL axiom $\xi \Rightarrow \forall x \xi$ (x is not free in ξ), which, as observed in Section 4, can be falsified if the language contains modalities.

For the purpose of illustrating the concept of fibring fob logics, we consider first the elementary example of generating a bi-modal first-order logic by fibring two uni-modal first-order logics.

Example 7.3 *2KFOL = KFOL′ ∪ KFOL″.* Let KFOL′ and KFOL″ be two copies of full KFOL (as described in Example 6.5) such that $\Sigma'(I, F, P)$ is identical to $\Sigma''(I, F, P)$ with the exception that $O'_1 = \{\Box'\}$ and $O''_1 = \{\Box''\}$. Each signature of the fibring is of the form $\Sigma'(I', F', P') \cup \Sigma''(I'', F'', P'')$ where the connectives $\neg$ and $\wedge$ are shared, as well as the quantifier $\forall$, but where we find two modalities ($\Box'$ and $\Box''$). For each such a signature in the fibring, a model is a structure whose reducts are structures corresponding to models in the given logics, and the sets of rules are obtained by the union of the corresponding sets of rules in the given logics. △

We now turn our attention to a more complex example where we obtain KFOL as the fibring of pure first-order logic and modal logic enriched with variables, individual symbols, equality and inequality.

Example 7.4 *KFOL as a fibring.* The idea is to obtain KFOL by fibring first-order logic and K modal propositional logic. To this end, first we have to present these two logics as fob logics. Presenting first-order logic as a fob logic is straightforward. However, when defining a modal propositional logic as a fob logic we are compelled to include in the language variables, as well as equalities and inequalities between them. So, we obtain a richer modal logic that nevertheless is quite appropriate to our objective. Indeed, in the richer modal logic, the entailments are the same for the original formulae. Furthermore, it is easy to establish a complete axiomatic system for the richer modal logic, given a complete axiomatic system for the original modal logic. Finally, it is straightforward to obtain KFOL by fibring first-order logic and the richer modal logic.

First-order logic. The fob logic FOL is easily defined as follows. Signatures are of the form

$$\Sigma(F, P) = \langle I, F, P, C, Q, O\rangle$$

in which $I = \emptyset$, F, P, C and Q are as for $KFOL$, but $P_0 = \emptyset$, and there are no modalities (that is, $O_k = \emptyset$ for every k).

For each FOL signature $\Sigma(F, P)$, its Hilbert system is as follows: R_{d} and R_{Qp} are as in Example 5.7, $R_{\mathrm{Op}} = \emptyset$, and $R_{\mathrm{p}} = R_{\mathrm{Qp}}$.

For each FOL signature $\Sigma(F, P)$, in its interpretation system $\langle M, A\rangle$ we let M be the class of all $\Sigma(F, P)$-structures appropriate for the Hilbert system at that signature and A be the identity map. Therefore, we obtain a full logic.

Modal logic. The fob modal logic KML^+ is defined as follows. Signatures are of the form

$$\Sigma(I,\Pi) = \langle I, F, P, C, Q, O\rangle$$

in which $F_k = \emptyset$ for every k, $P_0 = \Pi$, $P_k = \emptyset$ for every $k > 0$, $C_1 = \{\neg\}$, $C_2 = \{\wedge\}$, $C_k = \emptyset$ for every $k > 2$, $Q_k = \emptyset$ for every k (that is, there are no quantifiers), $O_1 = \{\Box\}$, and $O_k = \emptyset$ for every $k > 1$.

For each KML^+ signature $\Sigma(I,\Pi)$, its Hilbert system is as follows: R_{Op} and R_{d} are as in Example 5.7, but the latter without the congruence axioms (since there are no function symbols and no predicate symbols), $R_{\mathrm{Qp}} = \emptyset$, and $R_{\mathrm{p}} = R_{\mathrm{Op}}$.

For each KML^+ signature $\Sigma(I,\Pi)$, in its interpretation system $\langle M, A\rangle$ we let M be the class of all $\Sigma(I,\Pi)$-structures appropriate for the Hilbert system at that signature and A be the identity map. Again, we get a full logic.

Fibring. Finally, we are able to recover the fob logic KFOL by fibring FOL and KML^+. Each signature of the fibring is of the form $\Sigma(F,P)\cup\Sigma(I,\Pi)$ where the connectives $\neg$ and $\wedge$ are shared. Note how important it was to endow the logics with a full semantics in order to obtain the envisaged models in the fibring. Otherwise, in the fibring, the modal part might collapse into classical logic. $\triangle$

Remark 7.5 In each of the two examples above we obtained the fibring of two logics which were assumed to be endowed with semantics in the style of Section 3 and deductive system in the style of Section 5. This is known as the *homogeneous scenario* for fibring. Unfortunately, in general, we may be given two logics with quite different types of semantics and of deduction system. Therefore, what we need is first to prepare each of the given logics before making the fibring. This *preprocessing step* is essential until a theory of heterogeneous fibring can be developed. At the semantic level, the preprocessing can be conceptually very simple: for each of the native models of the logic we try to generate a structure in the sense of Definition 3.1 and try to prove that the entailments are preserved. At the deduction level, things can be much more complicated if no equivalent Hilbert calculus is known. $\triangle$

We conclude this section with a result comparing derivations in the given logics with derivations in the fibring. Since, given a derivation $\langle\varphi_1,\pi_1\rangle,\ldots,\langle\varphi_n,\pi_n\rangle$, for instance in $\mathcal{L}'$, of φ from Γ constrained by π, precisely the same sequence $\langle\varphi_1,\pi_1\rangle,\ldots,\langle\varphi_n,\pi_n\rangle$ constitutes a derivation in $\mathcal{L}$ of φ from Γ constrained by π, we have:

Proposition 7.6 In a fibring $\mathcal{L} = \mathcal{L}'\cup\mathcal{L}''$, if $\Gamma \vdash'_{\mathrm{d}} \varphi : \pi$ or $\Gamma \vdash''_{\mathrm{d}} \varphi : \pi$, then $\Gamma \vdash_{\mathrm{d}} \varphi : \pi$. $\triangle$

Remark 7.7 The meaning of this proposition is that $\mathcal{H}'\cup\mathcal{H}''$ is a conservative extension of the two given Hilbert frameworks. This result will be frequently used in the next section in the following way. Assume that a property of Hilbert calculi is equivalent to the fact that $\Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi : \pi$ for Γ and φ in a suitable class of formulae. Then, by Proposition 7.6, properties of this kind are trivially preserved. Most of the preservation results of the next section will be proved in this way.

8 Completeness preservation

In this section, we consider the problem of preservation of completeness by fibring. It turns out that completeness is indeed preserved by fibring under some natural assumptions that are fulfilled in a wide class of logics encompassing the most common fob logics. We establish the preservation result by invoking the Completeness Theorem (Theorem 6.6) proved in Section 6.

Proposition 8.10 below shows that the fibring of two full logics is still a full logic. Similarly, Proposition 8.9 shows that also uniformity is preserved by fibring. In general, however, the properties of logics considered in the Completeness Theorem are not always preserved by fibring. Example 5.8 in [17], for instance, provides two congruent Hilbert systems with non-congruent fibring.[3] The aim of the present section is to determine a class of logics which is closed under fibring and such that every element of it enjoys the properties considered in Theorem 6.6. As a corollary of this theorem, we will have that completeness is preserved by fibring logics from that class. In the first part of this section, we present and discuss particular properties of Hilbert frameworks, and hence of logics, which will be shown to be preserved by fibring. In the second part, we will show that Theorem 6.6 applies to logics with those properties.

Hilbert frameworks with implication and equivalence. Usual logics have implication and equivalence, and the properties presented in the following definitions are the minimal ones for logics with these connectives. In particular, Definition 8.1 relates implication with deduction, while Definition 8.2 relates equivalence with implication, and states that equivalence must be a congruence for connectives and operators.

Definition 8.1 (a) A Hilbert framework $\mathcal{H}$ is said to be a *Hilbert framework with implication* iff, for each signature $\Sigma \in Sig$, C_2 contains $\Rightarrow$ and the *Metatheorem of Modus Ponens* (MTMP) and the *Metatheorem of Deduction* (MTD) hold: For every p-deductively closed $\Gamma \subseteq L(\Sigma, X, \Theta, \Xi)$ and $\varphi_1, \varphi_2 \in L(\Sigma, X, \Theta, \Xi)$,

$$\frac{\Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi_1 \Rightarrow \varphi_2 : \pi}{\Gamma, \varphi_1 \vdash^{\Sigma}_{\mathrm{d}} \varphi_2 : \pi} \qquad \text{(MTMP)}$$

$$\frac{\Gamma, \varphi_1 \vdash^{\Sigma}_{\mathrm{d}} \varphi_2 : \pi}{\Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi_1 \Rightarrow \varphi_2 : \pi} \qquad \text{(MTD)}$$

(b) A logic $\mathcal{L}$ is said to be a *logic with implication* iff its Hilbert framework is with implication. △

[3] The definition of congruent system given in [17] is different from Definition 5.12, but the example can be easily adapted to the present case.

Definition 8.2 (a) A Hilbert framework $\mathcal{H}$ with implication $\Rightarrow$ is said to be a *Hilbert framework with equivalence* iff, for each signature $\Sigma \in Sig$, C_2 contains $\Leftrightarrow$ and the *Metatheorems of Biconditionality 1 and 2* (MTB1,2), and the *Metatheorems of Substitution of Equivalents 1-3* (MTSE1-3) hold: For every $\Gamma \subseteq L(\Sigma, X, \Theta, \Xi)$, $\varphi_1, \varphi_2, \varphi_i, \varphi_i' \in L(\Sigma, X, \Theta, \Xi)$, $c \in C_k$, $q \in Q_k$, and $o \in O_k$,

$$\frac{\Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi_1 \Rightarrow \varphi_2 : \pi \qquad \Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi_2 \Rightarrow \varphi_1 : \pi}{\Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi_1 \Leftrightarrow \varphi_2 : \pi} \tag{MTB1}$$

$$\frac{\Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi_1 \Leftrightarrow \varphi_2 : \pi}{\Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi_1 \Rightarrow \varphi_2 : \pi \qquad \Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi_2 \Rightarrow \varphi_1 : \pi} \tag{MTB2}$$

$$\frac{\Gamma \vdash^{\Sigma}_{\mathrm{d}} \varphi_i \Leftrightarrow \varphi_i' : \pi\,, \text{ for } i = 1, \ldots, k}{\Gamma \vdash^{\Sigma}_{\mathrm{d}} c(\varphi_1, \ldots, \varphi_k) \Leftrightarrow c(\varphi_1', \ldots, \varphi_k') : \pi} \tag{MTSE1}$$

$$\frac{\Gamma^{\vdash_{\mathrm{Qp}}} \vdash^{\Sigma}_{\mathrm{d}} \varphi_i \Leftrightarrow \varphi_i' : \pi\,, \text{ for } i = 1, \ldots, k}{\Gamma^{\vdash_{\mathrm{Qp}}} \vdash^{\Sigma}_{\mathrm{d}} qx(\varphi_1, \ldots, \varphi_k) \Leftrightarrow qx(\varphi_1', \ldots, \varphi_k') : \pi} \tag{MTSE2}$$

$$\frac{\Gamma^{\vdash_{\mathrm{Op}}} \vdash^{\Sigma}_{\mathrm{d}} \varphi_i \Leftrightarrow \varphi_i' : \pi\,, \text{ for } i = 1, \ldots, k}{\Gamma^{\vdash_{\mathrm{Op}}} \vdash^{\Sigma}_{\mathrm{d}} o(\varphi_1, \ldots, \varphi_k) \Leftrightarrow o(\varphi_1', \ldots, \varphi_k') : \pi} \tag{MTSE3}$$

(b) A logic $\mathcal{L}$ is said to be a *logic with equivalence* iff its Hilbert framework is with equivalence. △

The definition of logic with equivalence given in [17] is different from the previous one. The proof of Proposition 6.5 in that paper, however, can be easily adapted to prove the second part of the next preservation result; the first part is Proposition 6.2 in [17].

Proposition 8.3 (a) The fibring of logics with implication is a logic with implication, provided that implication is shared at each signature. (b) The fibring of logics with equivalence is a logic with equivalence, provided that both implication and equivalence are shared at each signature. △

Persistent Hilbert frameworks. Horizontally and vertically persistent Hilbert frameworks were considered and motivated in Section 5. Definition 8.4 and Lemma 8.16 below provide a sufficient condition for horizontal and vertical persistence.

Definition 8.4 (a) We say that a Hilbert framework is *Qp-persistent* if, for every rule $r = \langle \Gamma, \eta, \pi \rangle \in R_{\mathrm{Qp}} \setminus R_{\mathrm{d}}$, either $\Gamma = \emptyset$, or

1. $\Gamma = \{\gamma\}$;
2. η is of the form $r(\gamma)$;
3. $\gamma \vdash_{\mathrm{d}} r(\gamma) : \mathrm{cfo}(\gamma)$;
4. for every rule $\langle \{\gamma_1, \ldots, \gamma_k\}, \gamma, \pi' \rangle$ in R_{d} $(k > 0)$, $\{r(\gamma_1), \ldots, r(\gamma_k)\} \vdash_{\mathrm{d}} r(\gamma) : \pi'$.

Op-persistent Hilbert frameworks are defined in the same way, by replacing R_{Qp} by R_{Op}, and $\mathrm{cfo}(\gamma)$ by $\mathrm{rig}(\gamma)$ in Clause 2.

(b) A logic $\mathcal{L}$ is said to be a *Qp-persistent logic* or a *Op-persistent logic* iff so is its Hilbert framework. △

The properties of Qp-persistence and of Op-persistence are generalizations of usual properties of first-order and of modal logics. The first-order rule of generalization, for instance has the properties 1 to 4 of the previous definition, where $r(\gamma)$ is of course $\forall x \gamma$.

Proposition 8.5 Qp-persistence and Op-persistence are preserved by fibring.

Proof: Clauses 1 and 2 for Qp- and Op-persistence hold in the fibring by Definition 7.2. Clauses 3 and 4 are expressed in the form outlined in Remark 7.7; thus, according to this remark, the preservation of these properties is a consequence of Proposition 7.6. QED

Hilbert frameworks with strong equality. The only difference between the requirements for equality and those for strong equality is that Clause 5 in Definition 5.13 is replaced in the latter by a requirement in which only rules (and not inferences) are involved. By Proposition 7.6, this will make preservations results almost straightforward.

Definition 8.6 A Hilbert system with implication is said to be with *strong equality* if Clauses 1 to 4 in Definition 5.13 are fulfilled and, in addition,

$$r_0 \;=\; \langle \{\theta = x \Rightarrow \xi\}, \xi, \pi \rangle \;\in\; R_{\mathrm{p}} \tag{8.1}$$

where, for each signature Σ, $\pi_\Sigma(\rho) = 1$ iff x does not occur in $\xi\rho$.

Lemma 8.7 In any logic, Clause 4. (i) of Definition 5.13 holds iff

(a) $t_1 = t_1', \ldots, t_k = t_k' \;\vdash_{\mathrm{d}}^{\Sigma}\; f(t_1, \ldots, t_k) = f(t_1', \ldots, t_k')\,,$

and, in logics with equivalence, Clause 4. (ii) of Definition 5.13 holds iff

(b) $t_1 = t_1', \ldots, t_k = t_k' \;\vdash_{\mathrm{d}}^{\Sigma}\; p(t_1, \ldots, t_k) \Leftrightarrow p(t_1', \ldots, t_k')\,.$

Proof: Clause 4. (i) of Definition 5.13 implies (a) for $\Gamma = \{t_1 = t_i' : i = 1, \ldots, k\}$ and $\pi = \mathbf{1}$. Assume now (a) and $\Gamma \vdash_{\mathrm{d}}^{\Sigma} t_i = t_i' : \pi$, for $i = 1, \ldots, k$. Then, we can clearly construct a proof of $f(t_1, \ldots, t_k) = f(t_1', \ldots, t_k')$ from Γ by suitably "putting together" the proofs of $t_i = t_i'$ from Γ and the proof of $f(t_1, \ldots, t_k) = f(t_1', \ldots, t_k')$ from the equalities $t_i = t_i'$. Since each of the former proofs is constrained by π and the latter is constrained by $\mathbf{1}$, we eventually obtain $\Gamma \vdash_{\mathrm{d}}^{\Sigma} f(t_1, \ldots, t_k) = f(t_1', \ldots, t_k') : \pi$.

By the properties of implication and of equivalence, Clause 4. (ii) is equivalent to

$$\frac{\Gamma \vdash_{\mathrm{d}}^{\Sigma} t_i = t_i' : \pi \text{ for } i = 1, \ldots, k}{\Gamma \vdash_{\mathrm{d}}^{\Sigma} p(t_1, \ldots, t_k) \Leftrightarrow p(t_1', \ldots, t_k') : \pi}$$

and hence the proof of that Clause 4. (ii) is equivalent to (b) is quite similar to that above. QED

Proposition 8.8 Strong equality is preserved by fibring Hilbert logics with equivalence, provided that both implication and equivalence are shared at each signature.

Proof: Use Proposition 7.6, and the remark below it, order to have that Clauses 1-3 hold in the fibring. The same proposition and Lemma 8.7 can be used in order to have that Clause 4 is preserved. The fibring fulfills (8.1) simply because of Definition 7.2. QED

This concludes the presentation of the classes of logics that will be involved in the main preservation result. The next two propositions provide other two partial results of this kind.

Proposition 8.9 Uniformity is preserved by fibring.

Proof: Straightforward from Definition 5.6 and Remark 7.7. QED

Proposition 8.10 Fullness is preserved by fibring.

Proof: We have to show that every $(\Sigma' \cup \Sigma'')$-structure s appropriate for $H(\Sigma' \cup \Sigma'')$ is in $A(M)$. That is, we have to show that $s|_{\Sigma'}$ is in $A'(M')$ and $s|_{\Sigma''}$ is in $A''(M'')$. Indeed, s is appropriate for both $H'(\Sigma')$ and $H''(\Sigma'')$, and, hence, $s|_{\Sigma'}$ is appropriate for $H'(\Sigma')$ and $s|_{\Sigma''}$ is appropriate for $H''(\Sigma'')$. Given the fullness of $\mathcal{L}'$ and $\mathcal{L}''$, $s|_{\Sigma'} \in A'(M')$ and $s|_{\Sigma''} \in A''(M'')$. QED

Finally, we can consider the properties involved in Theorem 6.6 and show that they hold in a suitable class of logics. The first result regards congruence; the following lemma can be proved in the same way as Theorem 6.6 in [17].

Lemma 8.11 Congruence holds in logics with equivalence.

This lemma and Proposition 8.3 provide a sufficient condition for the preservation of congruence:

Proposition 8.12 Congruence is preserved by fibring logics with equivalence, provided that both implication and equivalence are shared at each signature.

The next results concern the preservation of the properties of equality.

Proposition 8.13 Every uniform Hilbert system with strong equality is a Hilbert system with equality.

Proof: Assume $\Gamma, t = i \vdash_{\rm d}^{\Sigma} \varphi : \pi$, where the invariant i does not occur in the rules of $H(\Sigma)$ and $\pi(\rho) = 0$ whenever i occurs in $\Gamma\rho$ or in $\varphi\rho$. By compactness we also have $\{\gamma_1, \ldots, \gamma_k\}, t = i \vdash_{\rm d} \varphi : \pi$ for some $\{\gamma_1, \ldots, \gamma_k\} \subseteq \Gamma$. Moreover, since we are assuming that the Hilbert system is with implication, we also have

$$\vdash_{\rm d} t = i \Rightarrow \varphi* : \pi, \quad \text{for } \varphi^* = (\gamma_1 \Rightarrow (\gamma_2 \Rightarrow \ldots \Rightarrow (\gamma_k \Rightarrow \varphi) \ldots))$$

Since we are considering a uniform Hilbert system, Proposition 5.10 implies $\vdash_{\rm d} t = x \Rightarrow \varphi^* : \pi$, which implies in turn

$$\vdash_{\rm p} t = x \Rightarrow \varphi^* : \pi \qquad (*)$$

Consider now the rule r_0 for strong equality in (8.1), and write π_0 for the proviso in it. Given any substitution σ such that $\sigma(\theta) = t$ and $\sigma(\xi) = \varphi^*$, $(*)$ implies $\vdash_{\rm p} \varphi^* : \pi * \pi_0\sigma$ and $\vdash_{\rm d} \varphi^* : \pi * \pi_0\sigma$.

Given any Σ-substitution ρ, we have $\pi_0\sigma(\rho) = \pi_0(\sigma\rho) = 1$ iff i does not occur in $\sigma(\xi)\rho$, that is, iff i does not occur in $\varphi^*\rho$. Thus, since $\pi(\rho) = 0$ whenever i occurs in $\varphi^*\rho$, we have that $\pi = \pi * \pi_0\sigma$ and hence $\vdash_{\rm d} \varphi^* : \pi$.

Using the properties of implication again, we have $\{\gamma_1, \ldots, \gamma_k\} \vdash_{\rm d} \varphi : \pi$ and $\Gamma \vdash_{\rm d} \varphi : \pi$. QED

Proposition 8.14 Equality is preserved by fibring uniform logics with strong equality and equivalence.

Proof: By Proposition 8.13 and Proposition 8.8. QED

Proposition 8.15 Inequality is preserved by fibring logics for equality and implication, provided that implication is shared at each signature.

Proof: Indeed it is straightforward to verify that Clauses 1. and 2. of Definition 5.14, respectively, hold iff:

1. $t = t', t \neq t' \vdash_{\rm d}^{\Sigma} \varphi$;
2. $\vdash_{\rm d}^{\Sigma} (t = t' \Rightarrow \varphi) \Rightarrow ((t \neq t' \Rightarrow \varphi) \Rightarrow \varphi)$.

Again, using Proposition 7.6, we obtain the envisaged preservation. QED

Proposition 8.16 1) If a Hilbert framework is Qp-persistent, then it is vertically persistent. 2) If a Hilbert framework is Op-persistent, then it is horizontally persistent.

Proof: We prove only claim 1) since the proof of 2) is quite similar. Assume $\Gamma^{\vdash_{\rm p}}, \Psi \vdash_{\rm Qp} \varphi : \pi * {\rm cfo}(\Psi)$ and assume that the inference contains N applications of rules in $R_{\rm Qp} \setminus R_{\rm d}$; we show that the inference can be transformed into an inference with $N - 1$ applications of those rules.

Consider the first part $\langle \gamma_1, \pi_1 \rangle, \ldots, \langle \gamma_k, \pi_k \rangle$ of the inference of φ from $\Gamma^{\vdash_{\rm p}} \cup \Psi$ and assume that γ_k is obtained by a rule r in $R_{\rm Qp} \setminus R_{\rm d}$ and that no rules in

$R_{\mathrm{Qp}} \setminus R_{\mathrm{d}}$ were used before. Then, γ_k is $r(\gamma_j)$ for some $j < k$. Consider the sequence $\langle \gamma_1, \pi_1 \rangle, \langle r(\gamma_1), \pi_1 \rangle, \ldots \langle \gamma_{k-1}, \pi_{k-1} \rangle, \langle r(\gamma_{k-1}), \pi_{k-1} \rangle, \langle \gamma_k, \pi_k \rangle$, where each π_i contains $\mathrm{cfo}(\Psi)$; we show that each pair in it can be derived from $\Gamma^{\vdash_{\mathrm{p}}} \cup \Psi$. This will conclude the proof because γ_k is $r(\gamma_j)$.

If $\gamma_i \in \Gamma^{\vdash_{\mathrm{p}}}$, then $r(\gamma_i)$ is also in $\Gamma^{\vdash_{\mathrm{p}}}$. If $\gamma_i \in \Psi$, then $\Gamma^{\vdash_{\mathrm{p}}}, \Psi \vdash_{\mathrm{d}} r(\gamma_i)$, using Condition 3. in Definition 8.4. Assume that γ_i is derived by means of an instance $\langle \{\psi_1, \ldots, \psi_n\}, \gamma_i, \pi' \rangle$ of a rule in R_{d}. Then, using Condition 4. of Definition 8.4, we can conclude that $\{r(\psi_1), \ldots, r(\psi_n)\} \vdash_{\mathrm{d}} r(\gamma_i) : \pi'$, and so we have the result by induction. QED

Proposition 8.17 Persistence is preserved by fibring Qp-persistent and Op-persistent logics.

Proof: By Proposition 8.5 and Lemma 8.16. QED

Theorem 8.18 (*Completeness Preservation.*) Completeness is preserved when fibring full, uniform, Qp- and Op-persistent logics with implication, equivalence, strong equality and inequality, provided that both implication and equivalence are shared at each signature.

Proof: The previous results of this section show that: (1) the fibring of two logics with the properties considered in the present theorem is still a logic with these properties and (2) the condition for completeness stated in Theorem 6.6 are consequences of the properties considered in the present theorem. QED

9 Concluding remarks

We were able to extend the main results in [17] from the context of propositional based logics to the context of first-order based logics. This extension raised several definitional problems. At the model-theoretic level, a suitably general notion of interpretation structure was found as the basis for defining an algebraic semantics for fibrings of fob logics. At the proof-theoretic level, we had to deal with side constraints on inference rules and we had to revise the structure of Hilbert systems.

Proving the envisaged completeness theorem within the context of first-order based logics turned out to be much more complex than expected. Besides fullness and congruence, which were the only assumed properties of the logics in [17], some other key assumptions regarding the independent behaviour of quantifiers and modalities were found to be necessary. The proof was carried out using a variation of the Henkin method that took advantage of the assumed presence of equality and inequality in the logic.

Finally, reasonable sufficient conditions were identified for the preservation of completeness when fibring fob logics. These conditions define a wide class of such logics, encompassing many logics with quantifiers and modal or relevance components among others.

A fob logic as understood in this paper is endowed with a semantics of algebraic nature. One wonders if it will be possible to study fibring of other types of logics and still obtain completeness preservation. A first step in this direction for propositional based logics is given in [5] where results about fibring of non-truth-functional (such as paraconsistent logics) are established. In this context, it seems worthwhile to pursue the study of fibring of protoalgebraic and algebraizable logics in the sense of Blok and Pigozzi [3, 4].

A fob logic as defined in this paper is endowed with a deductive system in the style of a Hilbert calculus. Fibring of logics with other types of inference systems raise specific problems and deserve further work. Some results in this direction for propositional based logics are obtained in [13].

These two lines of research are still to be initiated within the context of first-order based logics. Dealing with higher-order quantification, as started in [7], is another obvious line of research.

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