Sup-lattice 2-forms and quantales*

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Abstract

A 2-form between two sup-lattices L and R is defined to be a sup-lattice bimorphism $L \times R \to 2$. Such 2-forms are equivalent to Galois connections, and we study them and their relation to quantales, involutive quantales and quantale modules. As examples we describe applications to C*-algebras.

Keywords: Sup-lattice, Galois connection, quantale, involutive quantale, quantale module, C*-algebra.

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1 Introduction

Let L and R be sup-lattices. A Galois connection between L and R is a pair of antitone maps $(-)^{\perp} : L \to R$ and ${}^{\perp}(-) : R \to L$ such that $x \leq {}^{\perp}(x^{\perp})$ and $y \leq ({}^{\perp}y)^{\perp}$ for all $x \in L$ and $y \in R$. In fact all the information present in the Galois connection is already available in each of the maps $(-)^{\perp}$ and ${}^{\perp}(-)$, due to completeness of the lattices, or equivalently in the map $\varphi : L \times R \to 2$ given by

$$\varphi(x,y) = 0 \iff x \leq {}^{\perp}y \quad (\iff y \leq x^{\perp}) \ ,$$

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which is a bimorphism of sup-lattices. In this paper we study such bimorphisms, and call them (sup-lattice) 2-forms. The purpose is to provide a useful framework within which to study various aspects of quantales and their modules, including involutive quantales and their applications to C*-algebras.

We study two notions of map between 2-forms: the orthomorphisms, in §3, which are analogous to isometries, and the continuous maps, in §4, so-named because they generalize the continuous maps of topological spaces. In particular, the set of continuous endomaps of a 2-form has the structure of a quantale, we show that under mild restrictions the 2-form can be recovered from it, and we obtain generalizations of well known facts [6] concerning the right and left sides of quantales of sup-lattice endomorphisms, and also concerning involutive quantales. In §5 we deal with principal quantale modules (i.e., modules with a single generator), and in §6 we relate them to 2-forms. Finally, in §7 we address the particular case of symmetric 2-forms and involutive quantales, and we discuss applications to C*-algebras.

We are indebted to the work of Mulvey and Pelletier [7], which was one of the main sources of inspiration for our paper. They implicitly use parts of the theory of 2-forms, and this is reflected in the fact that we obtain, in §7, a much shorter proof of one of their main theorems [7, Th. 9.1], which concerns the relation between quantales and C*-algebras. We hope in this way to bring out more explicitly some of the principles that lie behind that relation.

2 Background

In this section we present some basic facts, terminology and notation concerning sup-lattices, quantales and quantale modules, however without attempting to be complete. Further basic reading about sup-lattices and quantales can be found in the first chapters of the book by Rosenthal [13], and further references will be cited throughout this section.

By a *sup-lattice* is meant a partially ordered set S each of whose subsets $X \subseteq S$ has a join (supremum) $\bigvee X$ in S (hence, a sup-lattice is a complete lattice). By a *homomorphism* of sup-lattices $f : S \to T$ is meant a map that preserves arbitrary joins: $f(\bigvee X) = \bigvee\{f(x) \mid x \in X\}$, for all $X \subseteq S$. The greatest element $\bigvee S$ of a sup-lattice S (the top) is denoted by 1_S, or 1, and the least element $\bigvee \emptyset$ (the bottom) by 0_S, or 0. The two-element sup-lattice

$\{0,1\}$ is denoted by 2. The order-dual of a sup-lattice S, i.e., S with the order reversed, is denoted by S^{op}. A homomorphism of sup-lattices $f : S \to T$ is said to be *strong* if $f(1) = 1$, and *dense* if the condition $f(x) = 0$ implies $x = 0$ for all $x \in S$.

Any sup-lattice homomorphism $f : S \to T$ has a right adjoint $f_* : T \to S$, which preserves all the meets (infima) in T and is defined by

$$f_*(y) = \bigvee \{x \in S \mid f(x) \le y\} .$$

Equivalently, f_* is the unique monotone map that satisfies the condition

$$f(x) \le y \iff x \le f_*(y)$$

for all $x \in S$ and $y \in T$.

If S is a sup-lattice, and $j : S \to S$ is a closure operator on S, the set of fixed-points of j, $S_j = \{x \in S \mid j(x) = x\}$, is a sup-lattice whose joins are given by $\bigvee^j X = j(\bigvee X)$, and the map $j : S \to S_j$ that sends each $x \in S$ to $j(x)$ is a surjective sup-lattice homomorphism. Any quotient of a sup-lattice arises like this, up to isomorphism, for if $f : S \to T$ is a surjective sup-lattice homomorphism then $j = f_* \circ f$ is a closure operator on S, and $T \cong S_j$ [1, 13].

The category **SL** of sup-lattices is monoidal [1], and a semigroup in it is a *quantale*, *unital* if the semigroup is a monoid, *involutive* if the semigroup has an involution. A left (resp. right) *module* over a quantale Q is a left (resp. right) action in **SL**. The multiplication of two elements a and b in a quantale Q is denoted by $a \cdot b$; if the quantale is unital, its multiplicative unit is denoted by e_Q, or simply e; if the quantale is involutive, the involution assigns to each $a \in Q$ an element that is denoted by a^*. The action of an element $a \in Q$ on $x \in M$, where M is a left Q-module, is denoted by ax (or xa for a right Q-module), and the module is *unital* if $ex = x$ for all $x \in M$ (resp. $xe = x$ for a right module). An element a of a quantale is *left-sided* (resp. *right-sided*) if $1 \cdot a \le a$ (resp. $a \cdot 1 \le a$). An element which is both left- and right-sided is *two-sided*. The set of left-sided elements of a quantale Q is denoted by $\mathrm{L}(Q)$ (this is a right Q-module under multiplication), and the set (a left Q-module) of right-sided elements is denoted by $\mathrm{R}(Q)$. A *factor* is a quantale Q whose set of two-sided elements is $\{0,1\}$.

For any sup-lattice S the set $\mathcal{Q}(S)$ of sup-lattice endomorphisms of S is a unital quantale under the pointwise ordering, with multiplication given by composition, $f \cdot g = g \circ f$, and we have $\mathrm{L}(\mathcal{Q}(S)) \cong S$ and $\mathrm{R}(\mathcal{Q}(S)) \cong S^{\mathrm{op}}$ [6]. Explicitly, for a unital quantale Q we have $\mathrm{L}(Q) = 1 \cdot Q$ and $\mathrm{R}(Q) = Q \cdot 1$, and thus, for $\mathcal{Q}(S)$, a left-sided element is the same as a "constant" map for some $s \in S$,

$$c_s(x) = \begin{cases} s & \text{if } x \neq 0 \ , \\ 0 & \text{if } x = 0 \ , \end{cases}$$

and a right-sided element is an annihilator of some $s \in S$,

$$a_s(x) = \begin{cases} 1 & \text{if } x \not\leq s \ , \\ 0 & \text{if } x \leq s \ . \end{cases}$$

It also follows from this that $\mathcal{Q}(S)$ is a factor.

Another example of unital quantale, for any monoid M, is the powerset 2^M under pointwise multiplication:

$$X \cdot Y = \{xy \mid x \in X,\ y \in Y\} \ .$$

This construction is universal in the sense that for any unital quantale Q and any homomorphism of monoids $h : M \to Q$ there is a unique homomorphism of unital quantales $\bar{h} : 2^M \to Q$ such that $\bar{h}(\{x\}) = h(x)$ for all $x \in M$. Hence, any unital quantale is a quotient of one of the form 2^M, for some monoid M.

Quotients of quantales and modules can be described in terms of closure operators with additional properties: a *quantic nucleus* [13], or simply a *nucleus*, on a unital quantale Q is a closure operator j on Q such that for all $a, b \in Q$ we have $j(a) \cdot j(b) \leq j(a \cdot b)$; and a *nucleus* on a left Q-module M is a closure operator k on M such that for all $a \in Q$ and $x \in M$ we have $ak(x) \leq k(ax)$ (see [10] or [12, §2.5]). Given nuclei j and k as above, Q_j is a unital quantale with multiplication $(a, b) \mapsto a * b$ defined by $a * b = j(a \cdot b)$, and M_k is a left Q-module with action $(a, x) \mapsto a \bullet x$ defined by $a \bullet x = k(ax)$. The surjective maps $j : Q \to Q_j$ and $k : M \to M_k$ are respectively a homomorphism of unital quantales and a homomorphism of left Q-modules, and any quotient of quantales or of left Q-modules arises like this, up to isomorphism. If j and k are further related by the condition $j(a)x \leq k(ax)$, for all $a \in Q$ and $x \in M$, then M_k is also a left Q_j-module.

If R is a ring with unit then the map that sends each subset of R to the additive subgroup it generates is a nucleus on the unital quantale 2^R, and thus $\mathrm{Sub}(R)$, the set of additive subgroups of R, is a unital quantale with multiplication defined by

$$a \cdot b = \{r_1 s_1 + \cdots + r_n s_n \mid r_i \in a,\ s_i \in b\} \ .$$

The left-sided elements of $\mathrm{Sub}(R)$ are then the left ideals of R.

More generally, in the case of a unital k-algebra A with k an arbitrary commutative ring, the set $\mathrm{Sub}_k(A)$ of all the k-submodules of A is a unital quantale, and if A is a topological k-algebra then the set $\overline{\mathrm{Sub}}_k(A)$ of all the closed k-submodules of A is a unital quantale with multiplication defined by

$$a \cdot b = \overline{\{r_1 s_1 + \cdots + r_n s_n \mid r_i \in a,\ s_i \in b\}}\ ,$$

where $\overline{(-)}$ denotes topological closure.

We can obtain examples of modules in a similar way. If R is a ring and M is a left R-module then the set $\mathrm{Sub}(M)$ of additive subgroups of M is a left module over $\mathrm{Sub}(R)$, with action defined by

$$ax = \{r_1 m_1 + \cdots + r_n m_n \mid r_i \in a,\ m_i \in x\}\ .$$

A similar expression, but including closure for the topology, gives us a left $\overline{\mathrm{Sub}}_k(A)$-module $\overline{\mathrm{Sub}}_k(M)$, consisting of all the closed k-submodules of M, from any topological left A-module M over a topological k-algebra A.

If A is a complex C*-algebra with unit, the unital quantale $\overline{\mathrm{Sub}}_{\mathbb{C}}(A)$ is involutive, with involution obtained pointwise from the involution of A. This involutive quantale is denoted by $\mathrm{Max}\, A$ in [4, 5, 7, 8], where it plays the role of the "noncommutative maximal spectrum" of A. If H is a Hilbert space, its norm-closed linear subspaces can be identified with the projections on H, and we denote the sup-lattice $\overline{\mathrm{Sub}}_{\mathbb{C}}(H)$ by $\mathcal{P}(H)$. Any C*-algebra representation $\pi : A \to \mathcal{B}(H)$ of A on H makes H a topological left A-module, thus making $\mathcal{P}(H)$ a left $\mathrm{Max}\, A$-module.

Let L, R, and M be sup-lattices, and $(-) * (-) : L \times R \to M$ a sup-lattice *bimorphism*, i.e., a map that preserves joins in each variable (e.g., the multiplication $Q \times Q \to Q$ of a quantale Q, or the action $Q \times M \to M$ of Q on a left module M):

$$\begin{aligned} \left(\bigvee X\right) * y &= \bigvee\{x * y \mid x \in X\}\ ; \\ x * \left(\bigvee Y\right) &= \bigvee\{x * y \mid y \in Y\}\ . \end{aligned}$$

We will consistently use the following notation for the residuations associated to $*$ (i.e., the right adjoints to the homomorphisms $(-) * y$ and $x * (-)$), for each $x \in L$, $y \in R$, and $z \in M$:

$$\begin{aligned} z/y &= \bigvee\{x \in L \mid x * y \leq z\}\ , \\ x\backslash z &= \bigvee\{y \in R \mid x * y \leq z\}\ . \end{aligned}$$

Also, we define the following annihilators: $\mathrm{ann}(x) = x\backslash 0$, and $\mathrm{ann}(y) = 0/y$. Hence, we have

$$y \leq x\backslash z \iff x * y \leq z \iff x \leq z/y \;,$$
$$y \leq \mathrm{ann}(x) \iff x * y = 0 \iff x \leq \mathrm{ann}(y) \;,$$

and also the following (in)equalities: $(z/y)*y \leq z$, $\mathrm{ann}(y)*y = 0$, $x*(x\backslash z) \leq z$, $x * \mathrm{ann}(x) = 0$, $x \leq (x * y)/y$, $y \leq x\backslash(x * y)$, $((x * y)/y) * y = x * y$, $x * (x\backslash(x * y)) = x * y$.

3 2-forms and orthomorphisms

Let S be a sup-lattice. Since its dual, S^{op}, is order isomorphic to $\mathrm{hom}(S, 2)$ [1], any Galois connection between two sup-lattices L and R is uniquely determined by a sup-lattice homomorphism $L \to \mathrm{hom}(R, 2)$, which in turn is equivalent to a sup-lattice bimorphism $L \times R \to 2$ (because we have an order isomorphism $\mathrm{hom}(L \otimes R, M) \cong \mathrm{hom}(L, \mathrm{hom}(R, M))$ for any sup-lattices L, R, M [1]). Such bimorphisms are analogous to bilinear forms on ring modules, and provide us with a convenient alternative language for describing Galois connections.

Definition 3.1 Let L and R be sup-lattices. A map $\varphi : L \times R \to 2$ that preserves arbitrary joins in each variable is called a *2-form* between L and R, and we usually write $\langle x, y\rangle$ or $\langle x, y\rangle_\varphi$ instead of $\varphi(x, y)$. Two elements $x \in L$ and $y \in R$ are *orthogonal* if $\langle x, y\rangle = 0$, in which case we write $x \perp y$. The form is *dense on the right* if $1 \perp y$ implies $y = 0$, *dense on the left* if $x \perp 1$ implies $x = 0$, and *dense* if it is both dense on the right and on the left. The form is *faithful on the right* if $x = y$ whenever $\langle z, x\rangle = \langle z, y\rangle$ for all $z \in L$, *faithful on the left* if $x = y$ whenever $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in R$, and *faithful*, or *non-singular*, or a *duality*, if it is both faithful on the right and on the left. A 2-form $\varphi : S \times S \to 2$ is *symmetric* if $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in S$.

Density on the right is equivalent to requiring the sup-lattice homomorphism $\langle 1, -\rangle : L \to 2$ to be dense, which justifies our terminology. It is also equivalent to requiring $y = 0$ whenever $z \perp y$ for all $z \in L$, i.e., whenever $\langle z, y\rangle = \langle z, 0\rangle$ for all $z \in L$, which shows that faithfulness on the right is a stronger condition than density on the right. Of course, these two conditions would be equivalent if we were dealing with forms on ring modules, and equivalent to saying that a form is non-degenerate on the right. Hence, for sup-lattices there are two natural notions of non-degeneracy on the right. We shall need both of them, so we have decided to use a different word for each, and non-degeneracy for none in order to avoid ambiguity. Similar remarks apply to density and faithfulness on the left.

In view of these remarks it may seem surprising that we have defined non-singular to mean the same as faithful, since for ring modules a non-degenerate form is not necessarily non-singular, but (3.4) below shows that in the case of sup-lattices this identification is appropriate.

Definition 3.2 Let $\varphi : L \times R \to 2$ be a 2-form, $x \in L$ and $y \in R$. The *(right) orthogonal image* of x is the element $x^\perp \in R$ defined by

$$x^\perp = \bigvee \{ y \in R \mid x \perp y \} .$$

Similarly, the *(left) orthogonal image* of y is given by

$${}^\perp y = \bigvee \{ x \in L \mid x \perp y \} .$$

The correspondence between Galois connections and 2-forms can be summarized as follows:

Proposition 3.3 *For any 2-form between sup-lattices L and R, the orthogonal images $(-)^\perp : L \to R$ and ${}^\perp(-) : R \to L$ form a Galois connection, and any Galois connection between L and R is uniquely determined in this way by the 2-form whose orthogonality relation is given by*

$$x \perp y \iff x \le {}^\perp y \; (\iff y \le x^\perp) .$$

Furthermore, we have:

1. *The following are equivalent:*
   - *(a)* φ *is dense on the right;*
   - *(b)* $1^\perp = 0$*;*
   - *(c)* 0 *is the unique element* $y \in R$ *such that* ${}^\perp y = 1$*;*
2. *The following are equivalent:*

(a) φ *is dense on the left;*

(b) ${}^{\perp}1 = 0$*;*

(c) 0 *is the unique element* $x \in L$ *such that* $x^{\perp} = 1$*;*

3. *The following are equivalent:*

   (a) φ *is faithful on the right;*

   (b) $(-)^{\perp}$ *is surjective;*

   (c) ${}^{\perp}(-)$ *is injective;*

   (d) $({}^{\perp}y)^{\perp} = y$ *for all* $y \in R$*.*

4. *The following are equivalent:*

   (a) φ *is faithful on the left;*

   (b) $(-)^{\perp}$ *is injective;*

   (c) ${}^{\perp}(-)$ *is surjective;*

   (d) ${}^{\perp}(x^{\perp}) = x$ *for all* $x \in L$*.*

Corollary 3.4 *A 2-form between sup-lattices* L *and* R *is faithful if and only if* $(-)^{\perp}$ *(equiv.,* ${}^{\perp}(-)$*) is an antitone order isomorphism.*

Let us see some explicit examples of Galois connections in the language of 2-forms.

Example 3.5 Let X be a topological space, with topology τ_X. Then we define a 2-form between 2^X and τ_X by

$$S \perp U \iff S \cap U = \emptyset \ ,$$

which is faithful on the right and dense on the left. More generally, given any closure operator $j : L \to L$ on a sup-lattice L, the assignment $x \mapsto j(x)$ defines a surjective sup-lattice homomorphism $j : L \to L_j$ (cf. §2), and we may define a 2-form on $L \times {L_j}^{\mathrm{op}}$ by $x \perp y \iff x \leq y$ in L. Furthermore, this form is necessarily faithful on the right, and it is dense on the left if and only if $j(0) = 0$ (i.e., the closure is dense). In particular, if we take L to be 2^X and L_j to be the lattice of closed sets of X then ${L_j}^{\mathrm{op}} \cong \tau_X$ and we obtain the same as before.

Example 3.6 Let ρ be a binary relation between two sets S and T. Then we have a 2-form between 2^S and 2^T given by

$$X \perp Y \iff x\rho y \text{ for all } x \in X,\, y \in Y\;.$$

For instance, if $S = T$ and we take $x\rho y$ to be $x \neq y$, we obtain

$$X \perp Y \iff X \cap Y = \emptyset\;,$$

as in the topological example above.

Example 3.7 Let R be a commutative ring, M and N R-modules, and $f : M \times N \to R$ a bilinear form. Then f induces an orthogonality relation between M and N, with respect to which we can define a sup-lattice 2-form $\varphi : \mathrm{Sub}_R(M) \times \mathrm{Sub}_R(N) \to 2$ as in the previous example: for all submodules $X \subseteq M$ and $Y \subseteq N$, put $X \perp Y$ if and only if $f(x, y) = 0$ for all $x \in X$ and $y \in Y$. This sup-lattice 2-form is dense on the left (resp. on the right) if and only if the bilinear form f is non-degenerate on the left (resp. on the right).

Definition 3.8 Let $\varphi : L \times R \to 2$ and $\varphi' : L' \times R' \to 2$ be 2-forms. An *orthomorphism* $\varphi \to \varphi'$ is a pair of sup-lattice homomorphisms $f : L \to L'$ and $g : R \to R'$ such that for all $x \in L$ and $y \in R$ we have

$$\langle f(x), g(y)\rangle = \langle x, y\rangle\;.$$

If both f and g are surjective the orthomorphism (f, g) is said to be a *quotient orthomorphism*. In that case φ' is an *orthoquotient*, or simply *quotient*, of φ.

Proposition 3.9 *Let $\varphi : L \times R \to 2$ and $\varphi' : L' \times R' \to 2$ be 2-forms, and $(f, g) : \varphi \to \varphi'$ an orthomorphism. If g is surjective then (f, g) commutes with $(-)^\perp$ in the sense that*

$$g(x^\perp) = f(x)^\perp$$

for all $x \in L$. If furthermore f is strong then (f, g) preserves density on the right (i.e., φ' is dense on the right if φ is), and, if g is also dense, then φ is dense on the right if and only if φ' is dense on the right. Obvious dual statements apply to ${}^\perp(-)$ and density on the left if f and g are interchanged.

Proof. Assume that g is surjective, and let $x \in L$. Then,

$$g(x^{\perp}) = g(\bigvee\{y \mid x \perp y\}) = \bigvee\{g(y) \mid x \perp y\} =$$

$$= \bigvee\{g(y) \mid f(x) \perp g(y)\} = \bigvee\{z \in R' \mid f(x) \perp z\} = f(x)^{\perp} .$$

If furthermore f is strong and φ is dense on the right we have

$$1_{L'}{}^{\perp} = f(1_L)^{\perp} = g(1_L{}^{\perp}) = g(0) = 0 ,$$

i.e., φ' is dense on the right; and if g is also dense we have $1_L{}^{\perp} = 0$ if and only if $g(1_L{}^{\perp}) = 0$, and thus φ is dense on the right if and only if φ' is. The dual facts, with f and g interchanged, are proved in a similar way. ∎

Proposition 3.10 *Let $\varphi : L \times R \to 2$ and $\varphi' : L' \times R' \to 2$ be 2-forms, and $(f, g) : \varphi \to \varphi'$ an orthomorphism. If g is surjective and φ is faithful on the left then f is an order embedding, and if f is surjective and φ is faithful on the right then g is an order embedding.*

Proof. Assume that g is surjective and φ is faithful on the left, i.e., ${}^{\perp}(z^{\perp}) = z$ for all $z \in L$ [cf. (3.3)]. Then (f, g) preserves $(-)^{\perp}$, by the previous proposition, and thus we have, for all $x, z \in L$,

$$f(x) \leq f(z) \Rightarrow f(x) \leq {}^{\perp}(f(z)^{\perp}) \iff f(x) \leq {}^{\perp}(g(z^{\perp})) \iff f(x) \perp g(z^{\perp})$$

$$\iff x \perp z^{\perp} \iff x \leq {}^{\perp}(z^{\perp}) = z ,$$

i.e., f is an order embedding. For f surjective and φ faithful on the right everything is similar. ∎

Corollary 3.11 *Let φ and φ' be 2-forms, and $(f, g) : \varphi \to \varphi'$ a quotient orthomorphism. If φ is faithful then both f and g are order isomorphisms.*

Hence, the faithful 2-forms are "simple" in the sense that their only quotient orthomorphisms are isomorphisms. The next proposition states that from any 2-form we can always obtain a faithful one by means of a quotient, and corresponds to the fact that any Galois connection restricts to a dual isomorphism between the lattices of "closed elements".

Proposition 3.12 *Let $\varphi : L \times R \to 2$ be a 2-form, and let $f : L \to L$ and $g : R \to R$ be the closure operators defined by $x \mapsto {}^\perp(x^\perp)$ and $y \mapsto ({}^\perp y)^\perp$. Let $L' = {}^\perp R = \{{}^\perp(x^\perp) \mid x \in L\}$ and $R' = L^\perp = \{({}^\perp y)^\perp \mid y \in R\}$ be the corresponding quotients of L and R, and define a map $\varphi' : L' \times R' \to 2$ to be the restriction of φ to $L' \times R'$. Then φ' is a faithful 2-form and the pair (f, g) defines a (quotient) orthomorphism from φ to φ'.*

Proof. First we remark that for all $x \in L$ and $y \in R$ we have

$$x \perp y \iff y \leq x^\perp = ({}^\perp(x^\perp))^\perp \iff {}^\perp(x^\perp) \perp y \,.$$

Hence, for each subset $X \subseteq L'$ and each $y \in R$ we have

$${}^\perp((\bigvee X)^\perp) \perp y \iff \bigvee X \perp y \iff x \perp y \text{ for all } x \in X \,,$$

which shows that φ' preserves joins in the left variable, because the join of X in L' is ${}^\perp((\bigvee X)^\perp)$. Similarly, φ' preserves joins on the right and is thus a 2-form, obviously faithful, see (3.3). Finally, we also obtain, for all $x \in L$ and $y \in R$,

$$x \perp y \iff {}^\perp(x^\perp) \perp y \iff {}^\perp(x^\perp) \perp ({}^\perp y)^\perp \,,$$

which means (f, g) is an orthomorphism. ∎

Definition 3.13 We refer to the faithful 2-form φ' of the previous proposition as the *orthogonal quotient* of φ.

We conclude this section with the following proposition, which will not be needed elsewhere in this paper, but which can be regarded as the "soft" version of (3.10):

Proposition 3.14 *Let $\varphi : L \times R \to 2$ and $\varphi' : L' \times R' \to 2$ be 2-forms, and $(f, g) : \varphi \to \varphi'$ an orthomorphism. If g is strong and φ is dense on the left then f is dense. If f is strong and φ is dense on the right then g is dense.*

Proof. Assume that φ is dense on the left. If g is strong then we have

$$f(x) = 0 \Rightarrow f(x) \perp 1 \iff f(x) \perp g(1) \iff x \perp 1 \iff x = 0 \,,$$

i.e., f is dense. The second part of the proof is similar. ∎

4 Quantales and 2-forms

The multiplication of a quantale Q has the property that, for all $X \subseteq Q$ and $a \in Q$, $(\bigvee X)\cdot a = 0$ if and only if $x \cdot a = 0$ for all $x \in X$, and $a \cdot \bigvee X = 0$ if and only if $a \cdot x = 0$ for all $x \in X$. Hence, we obtain a 2-form from any quantale, as follows:

Definition 4.1 For any quantale Q, we define a 2-form $\Phi(Q)$ between $\mathrm{L}(Q)$ and $\mathrm{R}(Q)$ by putting, for each $a \in \mathrm{L}(Q)$ and $b \in \mathrm{R}(Q)$,

$$a \perp b \iff a \cdot b = 0 \ .$$

Now we study a converse to this, i.e., a way of obtaining a quantale from a 2-form, after which we relate the two constructions.

Definition 4.2 Let $\varphi : L \times R \to 2$ and $\varphi' : L' \times R' \to 2$ be 2-forms. A *continuous map* from φ to φ' is a pair (f, g) of contravariant sup-lattice homomorphisms, where $f : L \to L'$ and $g : R' \to R$, such that the following *continuity* condition is satisfied: for all $x \in L$ and $y \in R'$,

$$\langle f(x), y \rangle = \langle x, g(y) \rangle \ .$$

Example 4.3 The above terminology is justified as follows. Let X and Y be topological spaces, with topologies τ_X and τ_Y, let $f : X \to Y$ be a map (not necessarily continuous), and let $g : \tau_Y \to \tau_X$ be a sup-lattice homomorphism. Seeing X and Y as 2-forms as in (3.5), the pair $(\tilde{f}, g)$, where $\tilde{f}$ is the direct image map of f, $\tilde{f}(S) = \{f(x) \mid x \in S\}$, is a continuous map of 2-forms if and only if $f : X \to Y$ is a continuous map of topological spaces and $g = f^{-1}$.

A generalization of this situation can be obtained from any pair of closure operators j and j' on sup-lattices L and L', respectively. From these we obtain 2-forms on $L \times {L_j}^{\mathrm{op}}$ and $L' \times {L'_{j'}}^{\mathrm{op}}$, as in the second part of (3.5), and if $f : L \to L'$ is a sup-lattice homomorphism, then (f, g) is a continuous map of 2-forms if and only if f satisfies the condition $f \circ j \leq j' \circ f$ (i.e., f is continuous with respect to the closure operators) and g is the restriction to $L_{j'}$ of the right adjoint f_*—see (4.6) below.

Proposition 4.4 *The continuity condition is equivalent to each of the following:*

1. $g_*(x^\perp) = f(x)^\perp$ *for all* $x \in L$,

2. $f_*({}^\perp y) = {}^\perp g(y)$ *for all* $y \in R$,

3. $f(x) \perp g_*(x^\perp)$ *and* $x \perp g(f(x)^\perp)$ *for all* $x \in L$,

4. $f_*({}^\perp y) \perp g(y)$ *and* $f({}^\perp g(y)) \perp y$ *for all* $y \in R$.

Proof. 1. Continuity can be rewritten as

$$f(x) \perp y \iff x \perp g(y) ,$$

which in turn is equivalent to the condition

$$y \leq f(x)^\perp \iff g(y) \leq x^\perp ,$$

whose right-hand side is equivalent to $y \leq g_*(x^\perp)$.

2. This is similar to the previous case, once we rewrite the continuity condition as

$$f(x) \leq {}^\perp y \iff x \leq {}^\perp g(y) ,$$

since now the left hand side is equivalent to $x \leq f_*({}^\perp y)$.

3. From the first condition, continuity is equivalent to the conjunction

$$g_*(x^\perp) \leq f(x)^\perp \text{ and } g_*(x^\perp) \geq f(x)^\perp .$$

The inequality $g_*(x^\perp) \leq f(x)^\perp$ is equivalent to $f(x) \perp g_*(x^\perp)$, and the other inequality is equivalent to $x^\perp \geq g(f(x)^\perp)$, i.e., to $x \perp g(f(x)^\perp)$.

4. Similar to the previous case, now using the second condition. ∎

Corollary 4.5 *Let* $\varphi : L \times R \to 2$ *and* $\varphi' : L' \times R' \to 2$ *be 2-forms.*

1. *If* φ *is faithful on the right then for each sup-lattice homomorphism* $f : L \to L'$ *there is at most one sup-lattice homomorphism* $g : R' \to R$ *such that* (f, g) *is a continuous map of 2-forms* $\varphi \to \varphi'$.

2. *If* φ' *is faithful on the left then for each sup-lattice homomorphism* $g : R' \to R$ *there is at most one sup-lattice homomorphism* $f : L \to L'$ *such that* (f, g) *is a continuous map of 2-forms* $\varphi \to \varphi'$.

Proof. 1. φ is faithful on the right if and only if the map $(-)^\perp : L \to R$ is surjective. Hence, the first condition of the proposition, $g_*(x^\perp) = f(x)^\perp$ for all $x \in L$, completely determines the right adjoint g_*, and thus it determines g.
2. Similar, taking into account the second condition of the proposition. ∎

Notice that we do not state, e.g. in the first part of this corollary, that for every f there is a g such that (f, g) is continuous. The second part of (4.3) provides an example in which for only some f this holds, and the following proposition gives a necessary and suficient condition for such g to exist.

Proposition 4.6 *Let $\varphi : L \times R \to 2$ and $\varphi' : L' \times R' \to 2$ be 2-forms, both faithful on the right. Let also $f : L \to L'$ be a homomorphism. Then the following are equivalent:*

1. *There is a homomorphism $g : R' \to R$ such that (f, g) is continuous;*
2. *$f({}^\perp(x^\perp)) \leq {}^\perp(f(x)^\perp)$ for all $x \in L$ (i.e., f is continuous with respect to the closure operators ${}^\perp((-)^\perp)$.*

If in addition φ is faithful on the left, then for each $f : L \to L'$ there is exactly one $g : R' \to R$ such that (f, g) is continuous.

Proof. Assume that (1) holds. Then we have $f(x) \perp f(x)^\perp$, and

$$f(x) \perp f(x)^\perp \iff x \perp g(f(x)^\perp) \iff {}^\perp(x^\perp) \perp g(f(x)^\perp) \iff$$
$$\iff f({}^\perp(x^\perp)) \perp f(x)^\perp \iff f({}^\perp(x^\perp)) \leq {}^\perp(f(x)^\perp) \; .$$

Now assume that (2) holds. Write j for the closure operator ${}^\perp((-)^\perp)$ on L, and k for the similar closure on L'. Then (2) is the condition $f \circ j \leq k \circ f$. We shall prove that the image of the restriction of f_* to L'_k is contained in L_j. Indeed, this is equivalent to the condition that $j \circ f_* \circ k \leq f_* \circ k$, which holds because

$$f \circ j \leq k \circ f \iff j \leq f_* \circ k \circ f \Rightarrow j \circ f_* \circ k \leq f_* \circ k \circ f \circ f_* \circ k \leq f_* \circ k \; ,$$

where the latter inequality follows from the fact that $f \circ f_* \leq \mathrm{id}_{L'}$ and $k \circ k = k$. Hence, f_* defines a meet preserving map $L'_k \to L_j$. Due to right faithfulness of φ and φ' we have order isomorphisms $L_j \cong R^{\mathrm{op}}$ and $L'_k \cong R'^{\mathrm{op}}$, and thus $g : R' \to R$ can be defined by composing f_* with the isomorphisms. Finally, if φ is also faithful on the left we have ${}^\perp(x^\perp) = x$ for all $x \in L$, and thus f is trivially continuous with respect to ${}^\perp((-)^\perp)$. ∎

Clearly, continuous maps are closed under composition, and thus we obtain another category of 2-forms, which furthermore is sup-lattice enriched. In particular, then, the continuous endomaps of any 2-form form a unital quantale:

Definition 4.7 Let $\varphi : L \times R \to 2$ be a 2-form. The *quantale of* φ, denoted by $\mathcal{Q}(\varphi)$, is the quantale of continuous endomaps of φ, with $(f,g) \leq (f',g')$ if and only if $f(x) \leq f'(x)$ and $g(y) \leq g'(y)$ for all $x \in L$ and $y \in R$, and with multiplication given by $(f,g)\cdot(f',g') = (f' \circ f, g \circ g')$.

In the case of a symmetric 2-form φ we have $(f,g) \in \mathcal{Q}(\varphi)$ if and only if $(g,f) \in \mathcal{Q}(\varphi)$, and at once we remark:

Proposition 4.8 *Let $\varphi : L \times L \to 2$ be a symmetric 2-form. Then the quantale $\mathcal{Q}(\varphi)$ is involutive, with the involution given by $(f,g)^* = (g,f)$. Conversely, if Q is an involutive quantale then $\Phi(Q)$ is isomorphic to a symmetric 2-form, and $\mathcal{Q}(\Phi(Q))$ is involutive, with the involution given by $(f,g)^* = (g',f')$, where $f' : \mathrm{R}(Q) \to \mathrm{R}(Q)$ and $g' : \mathrm{L}(Q) \to \mathrm{L}(Q)$ are defined by $f'(y) = f(y^*)^*$ and $g'(x) = g(x^*)^*$.*

Example 4.9 Let us relate the quantales of endomorphisms of 2-forms to the well known endomorphism quantales of sup-lattices.

1. Let $\varphi : L \times R \to 2$ be a faithful 2-form. From (4.6) it follows that the quantales $\mathcal{Q}(\varphi)$ and $\mathcal{Q}(L)$ are isomorphic.

2. Let L be a sup-lattice, and define a 2-form $\varphi : L \times L^{\mathrm{op}} \to 2$ by $x \perp y$ if and only if $x \leq y$ in L (in other words, consider the Galois connection between L and L^{op} defined by the identity $(-)^\perp = \mathrm{id}_L : L \to (L^{\mathrm{op}})^{\mathrm{op}}$). This 2-form is faithful, and thus the quantales $\mathcal{Q}(\varphi)$ and $\mathcal{Q}(L)$ are isomorphic.

3. Let $\varphi : L \times L \to 2$ be both symmetric and faithful. Then $\mathcal{Q}(\varphi)$ is isomorphic to $\mathcal{Q}(L)$, which is thus involutive. The involution is defined on $\mathcal{Q}(L)$ in the usual way for quantales of endomorphisms on self-dual sup-lattices [6]:

$$f^*(y) = \left(\bigvee\{x \mid f(x) \leq y^\perp\}\right)^\perp .$$

Example 4.10 Kruml [3] defines a *Galois quantale* to be a quantale $\mathcal{Q}(G) = \{(f,g) \in \mathcal{Q}(S) \times \mathcal{Q}(T) \mid g \circ G = G \circ f\}$ for some sup-lattice homomorphism $G : S \to T$. From (4.4) it follows that Galois quantales are the same as quantales of 2-forms: $\mathcal{Q}(G)$ is isomorphic to $\mathcal{Q}(\varphi)$ for the 2-form $\varphi : S \times T^{\mathrm{op}} \to 2$ such that $(-)^\perp = G$.

Lemma 4.11 *Let $\varphi : L \times R \to 2$ be a dense two-form. Then $\mathrm{L}(\mathcal{Q}(\varphi))$ is order isomorphic to L, and $\mathrm{R}(\mathcal{Q}(\varphi))$ is order isomorphic to R. Furthermore $\mathcal{Q}(\varphi)$ is a factor quantale.*

Proof. First we remark that $\mathcal{Q}(\varphi)$ is a subquantale of $\mathcal{Q}(L) \times \mathcal{Q}^*(R)$, where $\mathcal{Q}^*(R)$ is the quantale $\mathcal{Q}(R)$ with reversed multiplication, i.e., with $f{\cdot}g = f \circ g$. Also, the top of $\mathcal{Q}(L) \times \mathcal{Q}^*(R)$ belongs to $\mathcal{Q}(\varphi)$ because φ is dense: for all $x \in L$ and $y \in R$, if either $x = 0$ or $y = 0$ then both conditions $1_{\mathcal{Q}(L)}(x) \perp y$ and $x \perp 1_{\mathcal{Q}^*(R)}(y)$ are true, whereas if $x \neq 0$ and $y \neq 0$ then both conditions are false. Hence, the left-sided elements of $\mathcal{Q}(\varphi)$ are precisely those which are left-sided as elements of $\mathcal{Q}(L) \times \mathcal{Q}^*(R)$, i.e., they are the continuous maps of the form (c_l, a_r) for some $l \in L$ and $r \in R$, where c_l and a_r are respectively a "constant" map and an annihilator, as described in §2. Hence, continuity means that for any pair of elements $x \in L$ and $y \in R$ we must have

$$\langle c_l(x), y\rangle = \langle x, a_r(y)\rangle \ .$$

Taking $x = 1$ yields $\langle l, y\rangle = \langle 1, a_r(y)\rangle$, and thus

$$\begin{aligned} l \perp y &\iff 1 \perp a_r(y) \\ &\iff a_r(y) = 0 \quad \text{(due to density)} \\ &\iff y \leq r \ . \end{aligned}$$

Therefore a necessary condition for continuity is $r = l^\perp$. The condition is also sufficient, since:

- if $x = 0$ then, trivially, $\langle c_l(x), y\rangle = \langle x, a_r(y)\rangle = 0$;
- if $x \neq 0$ then $c_l(x) = l$ and we obtain

$$\begin{aligned} \langle c_l(x), y\rangle = 0 &\iff l \perp y \\ &\iff y \leq r \\ &\iff a_r(y) = 0 \\ &\iff \langle x, a_r(y)\rangle = 0 \ , \end{aligned}$$

where the latter step follows from density on the left and the fact that $a_r(y)$ equals either 0 or 1. Hence, the generic form of a left-sided element of $\mathcal{Q}(\varphi)$ is $(c_l, a_{l^\perp})$, which means we have an assignment $l \mapsto (c_l, a_{l^\perp})$ that defines a surjective map $L \to \mathrm{L}(\mathcal{Q}(\varphi))$. Furthermore, we have $l \leq k$ if and only if $c_l \leq c_k$, and $l \leq k$ implies $a_{l^\perp} \leq a_{k^\perp}$, which makes the map $L \to \mathrm{L}(\mathcal{Q}(\varphi))$

an order-isomorphism. For right-sided elements the proof is analogous: each right-sided element of $\mathcal{Q}(\varphi)$ must be of the form (a_l, c_r) for some $l \in L$ and $r \in R$, and continuity is the condition

$$\langle a_l(x), y\rangle = \langle x, c_r(y)\rangle \ .$$

Taking again density of φ into account, we conclude that $l = {}^\perp r$, and thus the right-sided elements must be of the form $(a_{^\perp r}, c_r)$. Hence, R is isomorphic to $\mathrm{R}(\mathcal{Q}(\varphi))$. Finally, the only elements that are simultaneously left- and right-sided are those for which $(c_l, a_{l^\perp}) = (a_{^\perp r}, c_r)$, with $l \in L$ and $r \in R$. The only solutions are $(1, 1)$ and $(0, 0)$, corresponding respectively to $l = r = 1$ and $l = r = 0$, i.e., $\mathcal{Q}(\varphi)$ is a factor. ∎

Example 4.12 Let L be a sup-lattice, and φ the 2-form on $L \times L^{\mathrm{op}}$ of (4.9)-(2) [i.e., with $x \perp y \iff x \leq y$]. From the isomorphism $\mathcal{Q}(L) \cong \mathcal{Q}(\varphi)$ we immediately obtain the well known isomorphisms $\mathrm{L}(\mathcal{Q}(L)) \cong L$ and $\mathrm{R}(\mathcal{Q}(L)) \cong L^{\mathrm{op}}$ [6].

Theorem 4.13 *Let $\varphi : L \times R \to 2$ be a dense 2-form. Then φ and $\Phi(\mathcal{Q}(\varphi))$ are isomorphic 2-forms.*

Proof. All that we have to do is show that the isomorphisms of the previous lemma commute with the forms, i.e., that for all $l \in L$ and $r \in R$ we have $l \perp r$ if and only if, in $\mathcal{Q}(\varphi)$, the following condition holds,

$$(c_l, a_{l^\perp})\cdot(a_{^\perp r}, c_r) = (0, 0) \ ,$$

or, equivalently, if and only if the two following conditions hold: (i) $a_{^\perp r}\circ c_l = 0$ and (ii) $a_{l^\perp} \circ c_r = 0$. Since we have $c_l(1) = l$, condition (i) holds if and only if $a_{^\perp r}(l) = a_{^\perp r}(c_l(1)) = 0$, which is equivalent to $l \leq {}^\perp r$. Similarly, condition (ii) holds if and only if $a_{l^\perp}(r) = 0$, which is equivalent to $r \leq l^\perp$. Hence, both (i) and (ii) are equivalent to $l \perp r$. ∎

It is not in general true that for a quantale Q we have $Q \cong \mathcal{Q}(\Phi(Q))$, but there is always a *comparison homomorphism* $\kappa : Q \to \mathcal{Q}(\Phi(Q))$:

Proposition 4.14 *Let Q be a quantale, and let $a \in Q$. The right action of a on $\mathrm{L}(Q)$ and the left action of a on $\mathrm{R}(Q)$ jointly define a continuous endomap $((-)\cdot a, a\cdot(-))$ of the 2-form $\Phi(Q)$. The comparison homomorphism $\kappa : Q \to \mathcal{Q}(\Phi(Q))$ defined by $a \mapsto ((-)\cdot a, a\cdot(-))$ is a homomorphism of quantales, unital if Q is unital. If furthermore Q is involutive [in which case $\mathcal{Q}(\Phi(Q))$ is involutive, see (4.8)] then κ preserves the involution.*

Proof. The first part is immediate from the associativity of multiplication in Q, for:

$$\langle x\cdot a, y\rangle = 0 \iff (x\cdot a)\cdot y = 0 \iff x\cdot(a\cdot y) = 0 \iff \langle x, a\cdot y\rangle = 0 \,.$$

Now let us see that κ is a homomorphism of quantales. First, it preserves multiplication because composition of continuous maps of 2-forms is defined by $(f', g') \circ (f, g) = (f' \circ f, g \circ g')$, and thus for all $a, b \in Q$ the product $\kappa(a)\cdot\kappa(b)$ equals

$$\kappa(b) \circ \kappa(a) = (((-)\cdot a)\cdot b, a\cdot(b\cdot(-))) = ((-)\cdot(a\cdot b), (a\cdot b)\cdot(-)) = \kappa(a\cdot b) \,.$$

If Q is unital then $\kappa(e)$ is the unit of $\mathcal{Q}(\Phi(Q))$, and if Q is involutive we have $\kappa(a^*)(x, y) = (x\cdot a^*, a^*\cdot y) = ((a\cdot x^*)^*, (y^*\cdot a)^*) = \kappa(a)^*(x, y)$, see (4.8). ∎

The comparison homomorphism is injective if and only if, for all $a, b \in Q$, if $x\cdot a = x\cdot b$ and $a\cdot y = b\cdot y$ for all $x \in \mathrm{L}(Q)$ and $y \in \mathrm{R}(Q)$ then we have $a = b$. A quantale satisfying this condition is usually said to be *faithful* [2, 9, 11].

5 Principal quantale modules

Definition 5.1 Let Q be a quantale. A left Q-module M is *principal* if it has a *generator*, i.e., an element $x \in M$ such that $Qx = \{ax \mid a \in Q\} = M$. Similar definitions apply to right modules.

Some basic obvious properties of principal modules are the following:

Proposition 5.2 *Let Q be a quantale.*

1. *Any left Q-module quotient of a principal left Q-module is principal.*
2. *If M is a principal left Q-module then it is a left Q-module quotient of Q.*
3. *If Q is unital then M is a principal Q-module if and only if it is a left Q-module quotient of Q.*

Proof. 1. If $f : M \to N$ is a surjective homomorphism of left Q-modules and M has a generator x then $f(x)$ is a generator of N.
2. If M is a left Q-module with a generator x then the map $Q \to M$ defined by $a \mapsto ax$ is a surjective homomorphism of left Q-modules.
3. If Q is unital then e is a generator of itself as a module. The rest follows from the previous two. ∎

Definition 5.3 Let Q be a quantale, and M a left Q-module. An element $x \in M$ is *invariant* if $ax \leq x$ for all $a \in Q$ (equivalently, if $1_Q x \leq x$).

Hence, the left-sided elements of a quantale Q are the invariant elements of Q when Q is seen as a left module over itself.

Proposition 5.4 *Let Q be a quantale, M a left Q-module, and m an element of M. The following are equivalent:*

1. *$\uparrow m$ is a left Q-module quotient of M, with action defined by $(a, x) \mapsto ax \vee m$ and quotient projection $Q \to \uparrow m$ given by $x \mapsto x \vee m$.*
2. *$\downarrow m$ is a left Q-submodule of M.*
3. *m is an invariant element of M.*

Proof. $(1 \Leftrightarrow 3)$ Condition 1 holds if and only if the map $(-) \vee m : M \to M$ is a nucleus of left Q-modules. So assume that m is invariant. Then, for all $a \in Q$ and $x \in M$ we have

$$a(x \vee m) = ax \vee am \leq ax \vee m ,$$

i.e., $(-) \vee m$ is a nucleus. Now assume that $(-) \vee m$ is a nucleus. Then for all $a \in Q$ we have $am = a(0 \vee m) \leq a0 \vee m = m$, i.e., m is invariant.
$(2 \Leftrightarrow 3)$ $\downarrow m$ is a sub-sup-lattice, so it is a submodule if and only if it is closed for the action. Let then m be invariant, $x \in \downarrow m$, and $a \in Q$. Then $ax \leq am \leq m$. Let now $\downarrow m$ be a submodule. Then $am \in \downarrow m$ for all $a \in Q$, i.e., m is invariant. ∎

Example 5.5 Let R be a ring, and M a left R-module. Then $\mathrm{Sub}(M)$ is a left module over the quantale $\mathrm{Sub}(R)$, and an invariant element $N \in \mathrm{Sub}(M)$ is the same as a submodule of M. Given such a submodule N, it follows that $\mathrm{Sub}(N)$ is a left $\mathrm{Sub}(R)$-submodule of $\mathrm{Sub}(M)$ and coincides with $\downarrow N$, whereas $\uparrow N$, which is order-isomorphic to $\mathrm{Sub}(M/N)$, is also isomorphic as a left $\mathrm{Sub}(R)$-module when $\uparrow N$ is given the action of (5.4) and $\mathrm{Sub}(M/N)$ is given the action induced by the left R-module structure of M/N.

Proposition 5.6 *Let Q be a quantale, M a left Q-module, and $x \in M$. The map $(-)x : Q \to M$ sends left-sided elements to invariant elements, and the residuation $(-)/x : M \to Q$ sends invariant elements to left-sided elements. Furthermore, if x is a generator then the residuation also reflects left-sided elements back into invariant ones, i.e., $m \in M$ is invariant if and only if m/x is left-sided.*

Proof. If $a \in Q$ is left-sided then $1(ax) = (1a)x \le ax$, i.e., ax is invariant. Now assume m is invariant. We always have $(m/x)x \le m$, and thus $1(m/x)x \le 1m$. Since m is invariant we also have $1(m/x)x \le m$, which is equivalent to $1(m/x) \le m/x$, i.e., m/x is left-sided. Now assume that x is a generator. Then the map $(-)x$ is surjective, the inequality $(m/x)x \le m$ becomes the equality $m = (m/x)x$, and thus m is the invariant element to which $(-)x$ maps any left-sided element of the form m/x. ∎

Corollary 5.7 *Let Q be a quantale, M a left Q-module, and $x \in M$. Then* $\mathrm{ann}(x)$ *is left-sided.*

Proof. $\mathrm{ann}(x) = 0/x$ is left-sided because 0 is invariant. ∎

Proposition 5.8 *Let Q be a quantale, M a left Q-module, and $x \in M$ a generator. Then the quotient $(-)x : Q \to M$ factors through the quotient $(-) \vee \mathrm{ann}(x) : Q \mapsto \uparrow\mathrm{ann}(x)$ and a dense homomorphism $\varphi : \uparrow\mathrm{ann}(x) \to M$. Furthermore, φ restricts to an order isomorphism $M/x = \{m/x \mid m \in M\} \cong M$.*

Proof. For each $a \in Q$ we have $ax = ax \vee 0 = ax \vee \mathrm{ann}(x)x = (a \vee \mathrm{ann}(x))x$. Hence, we have $(a \vee \mathrm{ann}(x))x \le ax$, which is equivalent to $a \vee \mathrm{ann}(x) \le (ax)/x$, i.e., the closure operator $a \mapsto (ax)/x$ on Q is greater or equal to $a \mapsto a \vee \mathrm{ann}(x)$, and thus φ is just the restriction of $(-)x$ to $\uparrow\mathrm{ann}(x)$. It is dense because $ax = 0$ is equivalent to $a \le \mathrm{ann}(x)$, and M is isomorphic to M/x because φ is surjective. ∎

Corollary 5.9 *x is a generator of M if and only if $(\uparrow\mathrm{ann}(x))x = M$.*

In [7] a notion of *point* of a quantale is based on having "enough generators", whereas in other papers it is related only to a kind of irreducibility [2, 9, 11]. Since these notions have some importance, we devote the rest of this section to some simple results relating the two, although they are not needed in the rest of the paper.

Definition 5.10 Let Q be a quantale, and M a left Q-module. M is said to be *irreducible* if it has no invariant elements besides 0 and 1, and *everywhere principal* if for all non-zero $m \in M$ there is a generator $x \le m$.

Theorem 5.11 *Let Q be a quantale, and M a left Q-module. If either of the following two conditions holds, then M is irreducible:*

1. *M is everywhere principal.*

2. *M has a generator x such that* $\mathrm{ann}(x)$ *is a maximal left-sided element of Q.*

Proof. 1. For this it suffices to see that if $x \in M$ is any generator then 1_M is the only invariant above x. So assume that m is an invariant such that $x \leq m$, where x is a generator. Then $1_M = \bigvee M = \bigvee Qx = 1_Q x \leq 1_Q m \leq m$.
2. Let $m \in M$ be invariant. Then m/x is left-sided, by (5.6), and thus either $m/x = \mathrm{ann}(x)$ or $m/x = 1_Q$. But $m = (m/x)x$ because x is a generator, and thus $m = \mathrm{ann}(x)x = 0_M$ or $m = 1_Q x = \bigvee Qx = \bigvee M = 1_M$, i.e., M is irreducible. ∎

Remark 5.12 In [7] the *points* of a(n involutive) quantale Q are certain right Q-modules which are atomic as sup-lattices and whose atoms are generators, being thus everywhere principal (see also §7). In certain places in [7] the hypothesis that the module satisfies an additional condition known as *non-triviality* is assumed. Although formulated differently, this is equivalent to the requirement that $\mathrm{ann}(x)$ be a maximal right-sided element for some atom x, which itself implies irreducibility, by the previous theorem.

6 Quantale modules and 2-forms

Let Q be a quantale, $\varphi : L \times R \to 2$ a 2-form, and $h : Q \to \mathcal{Q}(\varphi)$ a homomorphism of quantales. For each $a \in Q$, the continuous endomap $h(a)$ is a contravariant pair of maps that defines a right action of Q on L and a left action of Q on R:

$$h(a) = ((-)a, a(-)) \; .$$

Examples of such homomorphisms are the comparison homomorphisms defined in §4, for which we have $\varphi = \Phi(Q)$. By definition of continuity the actions of Q on these modules satisfy, for all $x \in L$, $y \in R$, and $a \in Q$, the following "middle-linearity" condition:

$$\langle xa, y \rangle = \langle x, ay \rangle \; .$$

(In other words, the 2-form can be identified with a sup-lattice homomorphism from $L \otimes_Q R$, rather than just $L \otimes R$, to 2—for tensor products of sup-lattices see [1].) In this section we shall study such pairs of modules:

Definition 6.1 Let Q be a quantale, and $\varphi : L \times R \to 2$ a 2-form. An *action* of Q on φ consists of a right action of Q on L and a left action of Q on R,[1] such that for all $x \in L$, $y \in R$, and $a \in Q$ we have $\langle xa, y\rangle = \langle x, ay\rangle$. When the latter condition holds we say the 2-form is *balanced* (with respect to the Q-modules L and R), or that it is a *2-form over* Q. If Q is unital, the 2-form is *unital* if both L and R are unital modules.

Proposition 6.2 *Let Q be a quantale, L a right Q-module, R a left Q-module, and $\varphi : L \times R \to 2$ a sup-lattice 2-form. The following are equivalent:*

1. $a\backslash(x^\perp) = (xa)^\perp$ *for all* $x \in L$ *and* $a \in Q$,
2. $({}^\perp y)/a = {}^\perp(ay)$ *for all* $y \in R$ *and* $a \in Q$,
3. $xa \perp a\backslash(x^\perp)$ *and* $x \perp a((xa)^\perp)$ *for all* $x \in L$ *and* $a \in Q$,
4. $({}^\perp y)/a \perp ay$ *and* $({}^\perp(ay))a \perp y$ *for all* $y \in R$ *and* $a \in Q$,
5. φ *is a 2-form over* Q.

Proof. This is an immediate consequence of (4.4), for being a 2-form over Q is the same as the map $((-)a, a(-))$ being continuous, and $(-)/a$ and $a\backslash(-)$ are the right adjoints to $(-)a$ and $a(-)$, respectively. ∎

Now we introduce the notion of orthomorphism that is appropriate in the present context.

Definition 6.3 Let φ and φ' be 2-forms over a quantale Q. An *orthomorphism over* Q, or simply a *Q-orthomorphism*, is an orthomorphism $(f, g) : \varphi \to \varphi'$ such that f is a homomorphism of right Q-modules and g is a homomorphism of left Q-modules.

Balance is preserved by surjections, as follows:

Lemma 6.4 *Let Q be a quantale, and $\varphi : L \times R \to 2$ a 2-form over Q. Let also $\varphi' : L' \times R' \to 2$ be any 2-form, such that L' is a right Q-module, and R' is a left Q-module. Let $(f, g) : \varphi \to \varphi'$ be an orthomorphism such that both f and g are surjective Q-module homomorphisms (resp. right and left). Then φ' is balanced.*

[1] Our notation was suggested by the fact that L is the "left part" of the 2-form, and R is the "right part". While unfortunately this has led to L being a *right* module and R a *left* module, the notation is consistent with the fact that often L is $\mathrm{L}(Q)$ and R is $\mathrm{R}(Q)$ for some quantale Q.

Proof. Let $x' \in L'$, $y' \in R'$, and $a \in Q$. Due to surjectivity, there is $x \in L$ such that $x' = f(x)$, and $y \in R$ such that $y' = g(y)$. Hence,

$$\varphi'(x'a, y') = \varphi'(f(x)a, g(y)) = \varphi'(f(xa), g(y)) =$$

$$= \varphi(xa, y) = \varphi(x, ay) = \cdots = \varphi'(x', ay') \ . \ \blacksquare$$

Lemma 6.5 *Let Q be a quantale, and $\varphi : L \times R \to 2$ a 2-form over Q. Then the closure operators on L and R defined by $x \mapsto {}^{\perp}(x^{\perp})$ and $y \mapsto ({}^{\perp}y)^{\perp}$ are nuclei of Q-modules.*

Proof. We prove this only for ${}^{\perp}((-)^{\perp})$, as the other case is similar. Let $x \in L$ and $a \in Q$. The condition $xa \perp (xa)^{\perp}$ is always true and equivalent to $x \perp a((xa)^{\perp})$, which in turn is equivalent to $a((xa)^{\perp}) \leq x^{\perp} = ({}^{\perp}(x^{\perp}))^{\perp}$, i.e., to ${}^{\perp}(x^{\perp}) \perp a((xa)^{\perp})$. Finally, this is equivalent to $({}^{\perp}(x^{\perp}))a \perp (xa)^{\perp}$, i.e., $({}^{\perp}(x^{\perp}))a \leq {}^{\perp}((xa)^{\perp})$, which is precisely the statement that $({}^{\perp}(-))^{\perp}$ is a nucleus of right Q-modules. $\blacksquare$

Theorem 6.6 *Let Q be a quantale, $\varphi : L \times R \to 2$ a 2-form over Q, and $f : L \to L$ and $g : R \to R$ the closure operators defined by $x \mapsto {}^{\perp}(x^{\perp})$ and $y \mapsto ({}^{\perp}y)^{\perp}$. Let $L' = {}^{\perp}R = \{{}^{\perp}(x^{\perp}) \mid x \in L\}$ and $R' = L^{\perp} = \{({}^{\perp}y)^{\perp} \mid y \in R\}$ be the corresponding quotients of L and R, and let the 2-form $\varphi' : L' \times R' \to 2$ be the restriction of φ to $L' \times R'$. Then φ' is a 2-form over Q.*

[In other words, if φ is a 2-form over Q then its simple quotient, as defined in (3.12), is also a 2-form over Q.]

Proof. Corollary of the previous lemmas and (3.12). $\blacksquare$

Lemma 6.7 *Let Q be a quantale, and $n \in Q$. Define a map $\varphi_n : Q \times Q \to 2$ by*

$$\varphi_n(x, y) = 0 \iff x \cdot y \leq n \ .$$

Then φ_n is a 2-form, and it is balanced with respect to Q, seen both as a right and a left module over itself.

Proof. We have $\varphi_n(\bigvee_i x_i, y) = 0$ if and only if $\bigvee_i x_i y \leq n$, which holds if and only if $x_i \cdot y \leq n$ for all i, i.e., $\varphi_n(x_i, y) = 0$ for all i. A similar fact holds for joins on the right variable, and thus φ_n is a 2-form on $Q \times Q$. Furthermore, this 2-form is obviously balanced with respect to the actions of Q on itself, due to the associativity of the multiplication in Q. $\blacksquare$

Definition 6.8 Let Q be a quantale. A 2-form $\varphi : L \times R \to 2$ over Q is *principal* if both L and R are principal Q-modules.

Hence, if Q is unital and $n \in Q$ then φ_n is principal, and so is any of its quotients.

Definition 6.9 Let Q be a quantale, and $\varphi : L \times R \to 2$ a principal 2-form with generators $x \in L$ and $y \in R$. The *orthogonalizer* of x and y is defined to be

$$\mathrm{orth}(x, y) = \bigvee \{a \in Q \mid x \perp ay\} .$$

Notice that the following equivalences hold,

$$a \leq \mathrm{orth}(x, y) \iff x \perp ay \iff ay \leq x^{\perp} \iff a \leq (x^{\perp})/y ,$$

and thus $\mathrm{orth}(x, y) = (x^{\perp})/y$. Also, we have

$$x \perp ay \iff xa \perp y \iff xa \leq {}^{\perp}y ,$$

whence $\mathrm{orth}(x, y) = x\backslash({}^{\perp}y)$.

Theorem 6.10 *Let Q be a quantale, $\varphi : L \times R \to 2$ a principal 2-form over Q, and $n = \mathrm{orth}(x, y)$ for a pair of generators $x \in L$ and $y \in R$. Then there is a Q-orthoquotient $(f, g) : \varphi_n \to \varphi$.*

Proof. The map $f : Q \to L$ defined by $a \mapsto xa$ is a surjective right Q-module homomorphism, and the map $g : Q \to R$ defined by $b \mapsto by$ is a surjective left Q-module homomorphism. Finally, for all $a, b \in Q$ we have

$$\varphi_n(a, b) = 0 \iff a{\cdot}b \leq n \iff x \perp aby \iff xa \perp by \iff f(a) \perp g(b) ,$$

which shows that (f, g) is a Q-orthomorphism. ∎

In particular, this gives us a classification of the principal 2-forms over a unital quantale Q:

Corollary 6.11 *Let Q be a unital quantale. Then φ is a principal 2-form over Q if and only if it is a Q-orthoquotient of φ_n for some $n \in Q$.*

If furthermore the 2-forms are required to be faithful we obtain:

Corollary 6.12 *Let Q be a unital quantale. Then φ is a faithful principal 2-form over Q if and only if it is isomorphic to the orthogonal quotient [necessarily a Q-orthoquotient, by (6.6)] of φ_n for some $n \in Q$.*

We conclude this section relating these 2-forms with the upsegment modules of §5.

Lemma 6.13 *Let Q be a quantale, $n \in Q$, $r \in \mathrm{R}(Q)$, and $l \in \mathrm{L}(Q)$, such that $r \vee l \leq n$, and let $\psi : \uparrow r \times \uparrow l \to 2$ be the restriction of φ_n to $\uparrow r \times \uparrow l$. Then:*

1. *with the quotient module structures of $\uparrow r$ and $\uparrow l$, ψ is a 2-form over Q which furthermore is a quotient of φ_n, and the orthogonal quotient of φ_n factors through it;*
2. *if ψ is dense on the right (resp. left) then l (resp. r) is the greatest left-sided (resp. right-sided) element below n;*
3. *if Q is unital then ψ is dense on the right (resp. left) if and only if l (resp. r) is the greatest left-sided (resp. right-sided) element below n.*

Proof. 1. First we remark that the joins in $\uparrow r$ are precisely the same as in Q, except for the join of the empty set, which in $\uparrow r$ is r. Similarly, the joins in $\uparrow l$ are those of Q but with the empty join being l. Hence, for ψ to be a 2-form it suffices to verify that it satisfies $\psi(r, y) = \psi(x, l) = 0$ for all $y \in \uparrow l$ and $x \in \uparrow r$. But we have $\psi(r, y) = \varphi_n(r, y) = 0$ if and only if $r \cdot y \leq n$, which is true because r is right-sided: $r \cdot y \leq r \leq n$. Similarly, $\psi(x, l) = 0$ because l is left-sided and $l \leq n$, and we conclude that ψ is a 2-form. Since it is a quotient of φ_n, which is a 2-form over Q, ψ is also a 2-form over Q. Finally, the orthogonal quotient of φ_n is the least quotient of φ_n and thus factors through ψ.
2. Now assume that ψ is dense on the right, and let $a \leq n$ be a left-sided element of Q. Then $a \vee l \leq n$, and $1 \cdot (a \vee l) \leq n$, i.e., $\psi(1, a \vee l) = 0$ (this makes sense because $a \vee l \in \uparrow l$). Hence, since ψ is dense it follows that $a \vee l = l$, i.e., $a \leq l$, which shows that l is the greatest left-sided element below n. The situation with density on the left is similar.
3. Assume that Q is unital and that l is the greatest left-sided element below n. Let $x \in \uparrow l$ such that $1 \perp x$, i.e., such that $1 \cdot x \leq n$. Then $1 \cdot x \leq l$ because $1 \cdot x$ is left-sided, and thus $x \leq l$, i.e. $x = l$, because Q is unital and thus $x \leq 1 \cdot x$. This shows that ψ is dense on the right. Density on the left is handled similarly. ∎

Theorem 6.14 *Let Q be a quantale, and $\varphi : L \times R \to 2$ a principal 2-form over Q with generators $x \in L$ and $y \in R$. Then:*

1. *if φ is dense on the right (resp. left) then* $\mathrm{ann}(y)$ *(resp.* $\mathrm{ann}(x)$*) is the greatest left-sided (resp. right-sided) element below* $\mathrm{orth}(x, y)$*;*

2. *if Q is unital then φ is dense on the right (resp. left) if and only if* $\mathrm{ann}(y)$ *(resp.* $\mathrm{ann}(x)$*) is the greatest left-sided (resp. right-sided) element below* $\mathrm{orth}(x, y)$*.*

Proof. Let $n = \mathrm{orth}(x, y)$. From (6.10) it follows that φ is a Q-orthoquotient of φ_n, and (5.8) implies that this quotient factors through $(f, g) : \psi \to \varphi$, where the 2-form $\psi : \uparrow\mathrm{ann}(x) \times \uparrow\mathrm{ann}(y) \to 2$ is as in (6.13), and both f and g are surjective and dense. Hence, by (3.9) we conclude that φ is dense on the right if and only if ψ is, and similarly on the left. The result now follows from (6.13). ∎

7 Involutive modules

If Q is an involutive quantale and we have a 2-form $\varphi : L \times R \to 2$ where both L and R are left Q-modules, it still makes sense to define when it is that φ is balanced, for the involution makes L a right module: $xa = a^*x$. Hence, being balanced corresponds to the condition $\langle a^*x, y\rangle = \langle x, ay\rangle$ for all $x \in L$, $y \in R$, and $a \in Q$, or, equivalently, $\langle ax, y\rangle = \langle x, a^*y\rangle$. We will not pursue this in general, but rather study the particular situation where $L = R$ and the 2-form is symmetric. Since in this situation we have $(-)^\perp = {}^\perp(-)$, we shall write $(-)^\perp$ for both.

Definition 7.1 Let Q be an involutive quantale, M a left Q-module, and φ a symmetric 2-form on M. The pair (M, φ) (or just M, when no confusion may arise) is an *involutive (left) Q-module* if for all $a \in Q$ and $x, y \in M$ we have

$$\langle a^*x, y\rangle = \langle x, ay\rangle \ .$$

A *homomorphism* of involutive left Q-modules is a homomorphism of left Q-modules f such that (f, f) is an orthomorphism: $\langle f(x), f(y)\rangle = \langle x, y\rangle$. An *involutive right module* is defined analogously by the condition

$$\langle xa, y\rangle = \langle x, ya^*\rangle \ .$$

The fact that we have restricted to symmetric 2-forms allows us to use the fact (4.8) that $\mathcal{Q}(\varphi)$ is an involutive quantale:

Proposition 7.2 *Let Q be an involutive quantale, and $\varphi : M \times M \to 2$ a symmetric 2-form. There is a bijection between involutive left Q-module structures on (M, φ) and involution preserving homomorphisms from Q to $\mathcal{Q}(\varphi)$.*

Proof. A quantale homomorphism $h : Q \to \mathcal{Q}(\varphi)$ is the same as an action of Q on φ, with $h(a) = ((-)a, a(-))$. Hence, h preserves involution if and only if $((-)a^*, a^*(-)) = ((-)a, a(-))^* = (a(-), (-)a)$, i.e., if and only if $(-)a = a^*(-)$ for all $a \in Q$, i.e., if and only if (M, φ) is an involutive left Q-module. ∎

From here and (4.9) we see that in the case when the 2-forms involved are faithful the notion of involutive module corresponds precisely to that of involutive representation $Q \to \mathcal{Q}(S)$ of [7, 11]. In other words we have, as a corollary of (6.2):

Proposition 7.3 *Let Q be an involutive quantale, M a left Q-module, and φ a symmetric 2-form on M. Then (M, φ) is an involutive left Q-module if and only if $(a^*x)^\perp = a\backslash(x^\perp)$ for all $a \in Q$ and $x \in M$. In the case when φ is faithful this condition is equivalent to $a^*x = (a\backslash(x^\perp))^\perp$.*

All the previous definitions and results can be specialized to the case of involutive modules. We highlight just a few facts:

Proposition 7.4 *Let Q be an involutive quantale, m a left-sided element, and n a self-adjoint element such that $m \le n$. Then the left Q-module $\uparrow m$ is involutive, with the symmetric 2-form being defined by $a \perp b \iff a^* \cdot b \le n$.*

Proof. Immediate consequence of (6.13), because m^* is right-sided and $m^* \le n^* = n$. ∎

Proposition 7.5 *Let Q be an involutive quantale, M an involutive left Q-module, and $x \in M$. Then $\mathrm{orth}(x, x)$ is self-adjoint.*

Proof. For all $a \in Q$ we have:

$$a \le \mathrm{orth}(x, x) \iff x \perp ax \iff a^*x \perp x \iff x \perp a^*x \iff$$
$$\iff a^* \le \mathrm{orth}(x, x) \ . \ \blacksquare$$

Proposition 7.6 *Let Q be an involutive quantale, and M an involutive left Q-module with a generator $x \in M$. Then $\uparrow\mathrm{ann}(x)$ is an involutive left Q-module with the symmetric 2-form defined by $a \perp_x b \iff a^* \cdot b \leq \mathrm{orth}(x, x)$, and the map $\uparrow\mathrm{ann}(x) \to M$ defined by $a \mapsto ax$ is a surjective homomorphism of involutive left Q-modules.*

Proof. Let us just see that the map $a \mapsto ax$ is an orthomorphism:

$$a \perp_x b \iff a^* \cdot b \leq \mathrm{orth}(x, x) \iff x \perp a^* bx \iff ax \perp bx \ . \ \blacksquare$$

Corollary 7.7 *Let Q, M, and x be as in the previous proposition. If the symmetric 2-form of $\uparrow\mathrm{ann}(x)$ is faithful then $\uparrow\mathrm{ann}(x)$ and M are isomorphic as involutive left Q-modules.*

We conclude by pointing out a few immediate consequences of these results in the case of quantales associated to C*-algebras. Recall from §2 that if A is a unital C*-algebra then the set of all the closed linear subspaces of A is a unital involutive quantale $\mathrm{Max}\, A$. We then have:

Lemma 7.8 *Let A be a unital C*-algebra, m a maximal left-sided element of $\mathrm{Max}\, A$, and $n = m \vee m^*$. Then the symmetric 2-form on $\uparrow m$ determined by n is faithful.*

Proof. From C*-algebra theory we know that there is a unique pure state $\varphi : A \to \mathbb{C}$ whose kernel is n, and that the quotient $H = A/m$ is a Hilbert space with inner product defined by $\langle a + m, b + m \rangle = \varphi(b^* a)$. Hence, two vectors $a + m, b + m \in H$ are orthogonal if and only if $b^* a \in n$, i.e., $a^* b \in n$, and thus the isomorphism $f : \uparrow m \to \mathcal{P}(H)$ is an orthomorphism because, for all $c, d \in \uparrow m$, $f(c)$ is orthogonal to $f(d)$ in the lattice $\mathcal{P}(H)$ of closed linear subspaces of H if and only if $c^* \cdot d \leq n$. Therefore the 2-form on $\uparrow m$ is faithful because the 2-form on $\mathcal{P}(H)$ is. $\blacksquare$

Theorem 7.9 *Let A be a unital C*-algebra, and M an involutive left $\mathrm{Max}\, A$-module with a generator x. Assume also that $\mathrm{ann}(x)$ is a maximal left-sided element. Then M is isomorphic as an involutive left $\mathrm{Max}\, A$-module to $\uparrow\mathrm{ann}(x)$.*

Proof. Let $m = \mathrm{ann}(x)$. The topological left A-module structure of A/m makes $\mathcal{P}(A/m)$ a left $\mathrm{Max}\, A$-module (cf. §2). Furthermore, A/m is involutive in the sense that $\langle ax, y \rangle = \langle x, a^* y \rangle$, and from here it follows easily that

$\mathcal{P}(A/m)$ is involutive as a Max A-module. Hence, we have a surjective homomorphism $\uparrow m \to M$ of involutive left Max A-modules, which must be an isomorphism because the 2-form on $\uparrow m$ is faithful. ∎

Following the terminology of [7], let us define a *Hilbert representation* of Max A to be an involutive left Max A-module isomorphic to one of the form $\mathcal{P}(H)$ determined by a representation of A on H, in the manner of §2. Then we obtain:

Corollary 7.10 *Let A be a unital C^*-algebra, and M an involutive left* Max A*-module with a generator x. Assume also that* $\mathrm{ann}(x)$ *is a maximal left-sided element. Then M is a Hilbert representation determined by an irreducible representation of A.*

Proof. This follows from the previous results and the fact that for a maximal ideal m the quotient A/m defines an irreducible representation. ∎

The existence of a generator whose annihilator is a maximal left-sided element is, as was already mentioned in (5.12), equivalent to the property known as non-triviality in [7], and the above corollary corresponds to one of the implications in [7, Th. 9.1]. The main difference between the proof in [7] and what we have done above is that we have used 2-forms. Also, we have focused less on the properties of those elements of Max A known as "pure states" and instead more on the annihilators of the generators of principal Max A-modules, e.g., formulating non-triviality directly in terms of the annihilators. In [7, Th. 9.1] it is further assumed that the module M is an *algebraically irreducible* representation, i.e., that M is atomic as a sup-lattice and that each atom is a generator (equivalently, M is atomic and everywhere principal). Therefore our present formulation is more general, even though it is not so in an essential way because in the proof of [7, Th. 9.1] the extra conditions are not used. In [7] it is further conjectured that every algebraically irreducible representation of Max A is non-trivial.

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