In J.-Y. Beziau and W. A. Carnielli, editors.
Paraconsistency with No Frontiers.
Studies in Logic and Practical Reasoning, v.4.
Amsterdam: Elsevier Science, 2006, pp.301-311.

INEFFABLE INCONSISTENCIES

João Marcos

ABSTRACT: For any given consistent tarskian logic it is possible to find another non-trivial logic that allows for an inconsistent model yet completely coincides with the initial given logic from the point of view of their associated single-conclusion consequence relations.

A paradox? This short note shows you how to do it.

This may be read as the description of an expedition into unexplored regions of abstract logic, the theory of valuations and paraconsistency.

1 INCONSISTENT CLASSICAL LOGIC

Plus on voit ce monde, et plus on le voit plein de contradictions et d'inconséquences.
—Voltaire, *Dictionnaire Philosophique*, XVIII century.

Take your preferred presentation of classical propositional logic. More concretely, take some denumerable set At of atomic sentences and some non-empty functionally complete set of logical constants C. As usual, the set $\mathcal{S}$ of classical formulas will be inductively built as the free algebra generated by C over At. Let $\mathcal{V}$ be a set of truth-values, $\mathcal{D} \subseteq \mathcal{V}$ a set of designated values (shades of truth) and $\mathcal{U} \subseteq \mathcal{V}$ a set of undesignated values (shades of falsehood), where $\mathcal{D} \cup \mathcal{U} = \mathcal{V}$ and $\mathcal{D} \cap \mathcal{U} = \varnothing$. Semantically, a classical state of the world will be simulated by an assignment $\mathsf{Asg} : \mathsf{At} \to \mathcal{V}$, where both $\mathcal{D}$ and $\mathcal{U}$ are required to be non-empty —usually, they are taken to be singletons, symbolizing 'the true' and 'the false', if you like. Yes, if you have a boolean mind, you will probably be expecting each such assignment Asg to be uniquely extendable into a valuation $\S : \mathcal{S} \to \mathcal{V}$, according to the truth-functional interpretation of each connective in C. Indeed, say you are talking about disjunction and negation, $\vee$ and $\sim$. In that case you are probably expecting their semantical interpretations to be induced by the set Sem of all valuations $\S : \mathcal{S} \to \mathcal{V}$ such that:

$$
\begin{aligned}
\S(\alpha \vee \beta) \in \mathcal{D} \quad &\text{iff} \quad \S(\alpha) \in \mathcal{D} \text{ or } \S(\beta) \in \mathcal{D} \\
\S(\sim\alpha) \in \mathcal{D} \quad &\text{iff} \quad \S(\alpha) \in \mathcal{U}
\end{aligned}
$$

Because classical logic has a truth-functional semantics and because this semantics was formulated above in order to display the dependence of each complex classical formula on its immediate subformulas, and only on them, each of the $\S$-clauses regulating the set Sem could be written with an 'iff' and have a very specific format, indicating the similarity between the algebra of classical formulas and the classical (boolean) algebra of truth-values.

The canonical single-conclusion tarskian consequence relation induced by Sem, denoted by $\models^{\mathsf{s}}_{\mathsf{Sem}} \subseteq \mathsf{Pow}(\mathcal{S}) \times \mathcal{S}$, is defined by:

$$(\text{Ent}^{\mathsf{s}}) \quad \Gamma \models^{\mathsf{s}}_{\mathsf{Sem}} \varphi \quad \text{iff} \quad \S(\Gamma) \not\subseteq \mathcal{D} \text{ or } \S(\varphi) \notin \mathcal{U}, \text{ for every } \S \in \mathsf{Sem},$$

where $\Gamma \cup \{\varphi\} \subseteq \mathcal{S}$.

Now, given any other set of formulas $\mathcal{S}$ and any set of truth-values $\mathcal{V}$, one can take $\mathsf{Sem}^\star$ as any set of valuations $\S : \mathcal{S} \to \mathcal{V}$, and the definition of $\models^{\mathsf{s}}_{\mathsf{Sem}^\star}$ as in (Ent$^{\mathsf{s}}$) will still make perfect sense, and it will define the consequence relation of *some* tarskian logic. With that idea in mind, theorists of valuations (cf. [9, 7, 6]) come and ask you to simply forget about the structure of the set of truth-values and concentrate on the set of valuations itself, whichever way it might be introduced. And that is precisely what we shall be doing from now on.

Let $\S^d : \mathcal{S} \to \mathcal{V}$ be an arbitrary mapping such that $\S^d(\varphi) \in \mathcal{D}$, for any $\varphi \in \mathsf{S}$. A valuation like this plays the role of an inconsistent model, making everything 'true' at once. Suppose you now build a set Sem^d by just adjoining $\S^d$ to the classical set of valuations Sem. Is the new associated single-conclusion consequence relation, $\models^{\mathsf{s}}_{\mathsf{Sem}^d}$, any different from the original consequence relation from classical logic? Surprising as it might seem, the answer is 'NO'. Indeed, suppose $\Gamma \models^{\mathsf{s}}_{\mathsf{Sem}} \varphi$, for some formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{S}$. In that case, $\S(\varphi) \in \mathcal{D}$ whenever $\S(\Gamma) \subseteq \mathcal{D}$, for any $\S \in \mathsf{Sem}$, by definition. This obviously still holds good for $\S^d$. Conversely, suppose $\Gamma \models^{\mathsf{s}}_{\mathsf{Sem}^d} \varphi$. Then, $\S(\varphi) \in \mathcal{D}$ whenever $\S(\Gamma) \subseteq \mathcal{D}$, for any $\S \in \mathsf{Sem}^d$. So, in particular, this holds good for every $\S \in \mathsf{Sem}$. Thus, Sem^d provides an alternative sound and complete semantics for classical logic in a single-conclusion formulation!

The literature on paraconsistent logics is prolific on vague definitions of the very phenomenon of paraconsistency, at all levels. It is not without some disquietness that we find in the paraconsistent jungle definitions such as:

- "Paraconsistent logics are non-trivial logics which can accomodate contradictory theories."
- "Paraconsistent logics are non-explosive logics."
- "Paraconsistent logics are logics having some inconsistent models."

From a semantical perspective, all such definitions tend to say, when properly formalized,[1] that the above alternative formulation of classical logic is paraconsistent. Yet it is characterized by the very same single-conclusion consequence relation of the first and more usual formulation of classical logic!

What's wrong, if anything?

[1] For such a formalization from the perspective of abstract logics, check for instance [4].

2 THE GENERAL RECIPE

My desire and wish is that the things I start with should be so obvious that you wonder why I spend my time stating them. This is what I aim at because the point of philosophy is to start with something so simple as not to seem worth stating, and to end with something so paradoxical that no one will believe it.
—Bertrand Russell, *The Philosophy of Logical Atomism*, 1918.

Again, take some set $\mathcal{S}$ of formulas built from the logical constants in some set C over the atomic sentences in At. As soon as we need below to talk about negation, we will simply suppose that there are schemas of the form $\sim\varphi$, where $\sim\in\mathsf{C}$, available for us. Next, take some set $\mathcal{D}$ of designated truth-values and some disjoint set $\mathcal{U}$ of undesignated truth-values. As usual, $\mathcal{V}=\mathcal{D}\cup\mathcal{U}$. Any set $\Gamma\subseteq\mathcal{S}$ will here be called a *theory*. In the previous section we talked about single-conclusion consequence relations. Given some set Sem of valuations $\S:\mathcal{S}\to\mathcal{V}$, you can also define the associated *canonical* MULTIPLE-CONCLUSION consequence relation $\models^{\mathsf{m}}_{\mathsf{Sem}}\subseteq\mathsf{Pow}(\mathcal{S})\times\mathsf{Pow}(\mathcal{S})$ (cf. [15]), by simply setting:

$$(\mathrm{Ent}^{\mathsf{m}})\quad \Gamma\models^{\mathsf{m}}_{\mathsf{Sem}}\Delta\quad\text{iff}\quad \S(\Gamma)\not\subseteq\mathcal{D}\text{ or }\S(\Delta)\not\subseteq\mathcal{U},\ \text{for every }\S\in\mathsf{Sem},$$

where $\Gamma\cup\Delta\subseteq\mathcal{S}$. Taking commas as unions and omitting curly braces, from an abstract viewpoint any *tarskian* consequence relation $\models$ defined as above will be characterized by the following universal axioms, where $\mathsf{Ptn}(\Sigma)$ denotes the set of all quase partitions[2] of the set Σ:

$$
\begin{array}{lll}
(\mathrm{C1}) & (\Gamma,\varphi\models\varphi,\Delta) & (\text{overlap})\\
(\mathrm{C2}) & (\forall\langle\Sigma_1,\Sigma_2\rangle\in\mathsf{Ptn}(\Sigma))(\Gamma,\Sigma_1\models\Sigma_2,\Delta)\ \ /\ \ (\Gamma\models\Delta) & (\text{cut})\\
(\mathrm{C3}) & (\Gamma\models\Delta)\ \ /\ \ (\Gamma',\Gamma\models\Delta,\Delta') & (\text{dilution})
\end{array}
$$

To be sure, a strong adequacy theorem connects canonical consequence relations and tarskian consequence relations: A consequence relation is characterizable by $(\mathrm{Ent}^{\mathsf{m}})$ if and only if it satisfies axioms (C1), (C2) and (C3) (check [15]). Having said that, I will from this point on assume that every logic has an associated consequence relation, though not necessarily a canonical / tarskian one. Each particular consequence relation is intended to embody some specific notion of inference, a collection of directives about what-follows-from-what.[3]

[2] A quasi partition is just like a partition, but the partition classes may be empty.

[3] An authoritative referee has called my attention to the alleged 'mistake' of calling 'tarskian' the class of logics whose consequence relation is *multiple-conclusion* and is axiomatized through clauses (C1)–(C3). He claimed that this is "what the literature calls 'Scott Consequence Relations'", and advised me to "see e.g. Gabbay's book on intuitionistic logic". Well, if there is a mistake involved in my decision, it is certainly not *my* mistake, and maybe not even of Gabbay's book (which book?). Dana Scott has indeed been one of the foremost authors to propose the study of multiple-conclusion versions of the preceding tarskian axioms, initially formulated only in terms of single-conclusion consequence relations, or rather, equivalently, in terms of consequence operators (check [21]). Typically, [12, 11, 13] are the papers published by Scott that are cited by those who claim that 'multiple-conclusion logics are scottian'. I know that too well —I have, elsewhere, made that confusion myself. Nonetheless, axiom (C2) is never to be found in those papers; at best one can find, in its place, the strictly weaker version of (C2) where Σ is a

There are of course some dumb examples of tarskian logics that you might prefer to avoid, for the sake of 'minimal enlightenment'. Given $\mathcal{S}$, $\mathcal{D}$ and $\mathcal{U}$, collect in $\mathsf{Sem}(\mathcal{D}) = \{\S : \S(\mathcal{S}) \subseteq \mathcal{D}\}$ all the valuations that are 'biased towards truth', and collect in $\mathsf{Sem}(\mathcal{U}) = \{\S : \S(\mathcal{S}) \subseteq \mathcal{U}\}$ all the valuations that are 'biased towards falsehood'. Any valuation $\S^d \in \mathsf{Sem}(\mathcal{D})$ will from now on be said to constitute a *dadaistic model*, and any valuation $\S^n \in \mathsf{Sem}(\mathcal{U})$ will be said to constitute a *nihilistic model*. Let Dada denote some non-empty subset of $\mathsf{Sem}(\mathcal{D})$, and let Nihil denote some non-empty subset of $\mathsf{Sem}(\mathcal{U})$. Obviously, in a logic having a non-empty set of designated values and a consequence relation characterized by Dada, every formula is a tautology, a thesis, a top particle; in a logic having a non-empty set of undesignated values and characterized by Nihil, every formula is an antilogy, an antithesis, a bottom particle; in a logic having both designated and undesignated values and characterized by models which are either dadaistic or nihilistic, any given formula follows from any other given formula; in a logic with no models, any given theory follows from any other given theory. We will call *overcomplete* any of the above four logics. If you have not seen this before, the surprising bit is that, while the distinctions are clearly visible if you use a multiple-conclusion abstract framework, the four paths to overcompleteness lead to only two different logics in a single-conclusion abstract framework.

For a quick summary, here are the names we will give to each of the above four kinds of overcompleteness, and the way they are characterized:

(1) **dadaistic logic**	(2) **nihilistic logic**	(3) **semitrivial logic**	(4) **trivial logic**
Semantical conditions:			
$\mathcal{D}_1 \neq \varnothing$	$\mathcal{U}_2 \neq \varnothing$	$\mathcal{D}_3 \neq \varnothing$ and $\mathcal{U}_3 \neq \varnothing$	—
$\mathsf{Sem}_1 = \mathsf{Dada}$	$\mathsf{Sem}_2 = \mathsf{Nihil}$	$\mathsf{Sem}_3 = \mathsf{Dada} \bigcup \mathsf{Nihil}$	$\mathsf{Sem}_4 = \mathsf{Dada} \bigcap \mathsf{Nihil}$
Single-conclusion abstract characterizations:			
$(\forall\beta\Gamma)$	$(\forall\alpha\beta\Gamma)$	$(\forall\alpha\beta\Gamma)$	$(\forall\beta\Gamma)$
$\Gamma \models^{\mathsf{s}}_1 \beta$	$\Gamma, \alpha \models^{\mathsf{s}}_2 \beta$	$\Gamma, \alpha \models^{\mathsf{s}}_3 \beta$	$\Gamma \models^{\mathsf{s}}_4 \beta$
Multiple-conclusion abstract characterizations:			
$(\forall\beta\Gamma\Delta)$	$(\forall\alpha\Gamma\Delta)$	$(\forall\alpha\beta\Gamma\Delta)$	$(\forall\Gamma\Delta)$
$\Gamma \models^{\mathsf{m}}_1 \beta, \Delta$	$\Gamma, \alpha \models^{\mathsf{m}}_2 \Delta$	$\Gamma, \alpha \models^{\mathsf{m}}_3 \beta, \Delta$	$\Gamma \models^{\mathsf{m}}_4 \Delta$

All four overcomplete logics are obviously tarskian (you can check, as an exercise, that they respect (C1), (C2) and (C3)). Moreover, if a logic is trivial then it is both dadaistic and nihilistic, and being either dadaistic or nihilistic a logic will also be semitrivial. As you should notice, $\models^{\mathsf{s}}_1 = \models^{\mathsf{s}}_4$, so the single-conclusion framework cannot *see* the difference between the situation in which all models

singleton. Scott's approach in the aforementioned papers, in fact, always seems quite tentative, and it shows no hint of a deep underlying semantic motivation. Not surprisingly, nowhere has Scott an adequacy theorem to offer about the weaker notion of consequence relation that he proposes. My own approach here, thus, cannot be 'scottian'. It is based instead on the work of Shoesmith & Smiley (cf. [14, 15, 22]).

satisfy all formulas and the situation in which the logic has no models. Even worse, $\models^{\mathsf{s}}_2 = \models^{\mathsf{s}}_3$, so single-conclusion consequence relations for which all formulas are always false are identical to consequence relations for which all formulas are either all false or all true. But perhaps we should agree that truth-blindness is a serious variety of blindness?

Single-conclusion truth-blindness and the upgraded multiple-conclusion consequence relation can help sorting out the paradox from the last section. Say that we have a *consistent logic* in case (i) the logic is non-dadaistic but (ii) every theory of the logic is derivable from the set of all of its formulas, that is, $\mathcal{S} \models^{\mathsf{m}} \Delta$, for every $\Delta \subseteq \mathcal{S}$. Clause (i) might be read as regulating the number of tautologies of our logic and clause (ii), stronger than (i), says that the set of all formulas cannot be compatibly sustained, all at once. This seems to meet our intuitions according to which varieties of inconsistency appear when 'too many things' are allowed to be true, in a logic. If one can count on dilution, (C3), the above definition implies that there is some β such that $\not\models^{\mathsf{m}} \beta$, and at the same time $\mathcal{S} \models^{\mathsf{m}}$, that is, $\mathcal{S} \models^{\mathsf{m}} \varnothing$. In that case, the addition of a dadaistic model to a consistent logic, as it was done in the last section, clearly gives place to inconsistency, once it occasions $\mathcal{S} \not\models^{\mathsf{m}}$. But in the single-conclusion case, given (C1), both semantics will deliver just the same: $\mathcal{S} \models^{\mathsf{s}} \varphi$, for every $\varphi \in \mathcal{S}$.

The situation gets particularly spiky when you think of a logic having a negation symbol $\sim$. Say that we have a *$\sim$-contradictory context* $\langle\Gamma, \Delta\rangle$, where $\Gamma \cup \Delta \subseteq \mathcal{S}$, in case there is some formula $\varphi \in \mathcal{S}$ such that both $\Gamma \models \varphi, \Delta$ and $\Gamma \models {\sim}\varphi, \Delta$; say that we have a *$\sim$-inconsistent model* $\S \in \mathsf{Sem}$ in case there is some formula $\varphi \in \mathcal{S}$ such that both $\S(\varphi) \in \mathcal{D}$ and $\S({\sim}\varphi) \in \mathcal{D}$. Given (C1) and a logic with a negation symbol, contradictory theories are unavoidable. The same does not happen, though, with inconsistent models —the usual set of models for classical logic and for other usual consistent logics does indeed avoid such anomalous models. Consistency of a logic $\mathcal{L}$ should of course be a presupposition for its *$\sim$-consistency*. Now, one further presupposition for a particular formulation of a logic $\mathcal{L}$ in terms of a semantics Sem to be called *$\sim$-inconsistent* is that Sem should have a $\sim$-inconsistent model.

Let's explain this again. Consider the following classical universal rules:

(R1)	$(\Gamma, \alpha, {\sim}\alpha \models \Delta)$	(*pseudo-scotus*, or *explosion*)
(R2)	$(\Gamma, \alpha, {\sim}\alpha \models \beta, \Delta)$	(*ex contradictione sequitur quodlibet*)

Obviously, (R1) implies (R2). Now, while the failure of *pseudo-scotus* corresponds to the existence of some $\sim$-inconsistent model (such as the dadaistic one), the failure of *ex contradictione* corresponds, more specifically, to the existence of some non-dadaistic $\sim$-inconsistent model (which is much more interesting). Yet the two rules will look exactly the same (as (R1) collapses into (R2)) inside a single-conclusion environment.

Summing up the above observations, we will from now on say that we are talking about a *$\sim$-consistent logic* in case this logic is non-dadaistic but it still does respect *pseudo-scotus* —thus, in case it has a canonical semantics, it will admit of no $\sim$-inconsistent model. Here then is the **Paradox of Ineffable Inconsistencies**:

Let $\mathcal{L}$ be any fixed non-overcomplete tarskian logic.
Then it is always possible to find an inconsistent logic $\mathcal{IL}$ such that:

$$\Gamma \models^{\mathsf{m}}_{\mathcal{IL}} \beta, \Delta \text{ iff } \Gamma \models^{\mathsf{m}}_{\mathcal{L}} \beta, \Delta \text{ (and, in particular, } \Gamma \models^{\mathsf{s}}_{\mathcal{IL}} \beta \text{ iff } \Gamma \models^{\mathsf{s}}_{\mathcal{L}} \beta),$$

yet:

$$\mathcal{S} \not\models^{\mathsf{m}}_{\mathcal{IL}} \text{ (while, by definition, } \mathcal{S} \models^{\mathsf{m}}_{\mathcal{L}} \Delta, \text{ for every } \Delta).$$

In case $\mathcal{L}$ has a symbol $\sim$ for negation and is $\sim$-consistent, then

$$\alpha, \sim\alpha \not\models^{\mathsf{m}}_{\mathcal{IL}} \text{ (while, by definition, } \Gamma, \alpha, \sim\alpha \models^{\mathsf{m}}_{\mathcal{L}} \Delta, \text{ for every } \Gamma \text{ and } \Delta).$$

You already know the simple strategy to make the above trick work: Just add to $\mathsf{Sem}_{\mathcal{L}}$ some dadaistic valuation. We will call the logic $\mathcal{IL}$ thus obtained the *inconsistent counterpart of* $\mathcal{L}$. As in the case of classical logic, in the last section, the inconsistent counterpart of a consistent logic is always identical to the original formulation of the logic from a single-conclusion perspective. But now we know that while the inconsistent counterpart of classical logic still validates rules such as *ex contradictione*, it does NOT validate *pseudo-scotus* any longer. Note that the paradox does not subsist if you add a nihilistic valuation instead of a dadaistic one. In that case you would need a single-premise multiple-conclusion framework for it to make sense.

The only conundrum we are left with is the following. Logics such as $\mathcal{IL}$ are very naturally obtained from their consistent counterparts, and they happen to be neither consistent nor, in general, overcomplete (at least $\mathcal{IL}$ is not overcomplete if the original logic $\mathcal{L}$ was not overcomplete either). Are we willing to call them *paraconsistent*?

3 PARACONSISTENCY IS NOT ENOUGH

To make advice agreeable, try paradox or rhyme.
—Mason Cooley, *City Aphorisms*, 14th Selection, 1994.

Universal logicians (cf. [2, 3, 1]) believe that logic should be seen a mother-structure (in the sense of Bourbaki) based on some given set of formulas and a consequence relation defined over it. They do not require in general these formulas and relation to bring any further built-in structure (say, an algebraic structure over the set of formulas). But in practical cases, of course, it is often interesting to fix for instance some set of axioms or another over the consequence relation. I have indeed presented above a multiple-conclusion version (cf. [15]) of the customary tarskian axioms (cf. [19]) and immediately after that I exhibited some trivial examples of tarskian logics: the overcomplete ones. Should we modify the given axioms in order to rule out these examples as illegitimate? One could surely do that, and it has indeed been done here and there in the literature, but I am not convinced that this is a very wise manoeuvre. First of all, the overcomplete logics fit very naturally both within the abstract and the semantical frameworks. Besides, I am only talking about 'overcomplete logics' once I had decided that they should be called 'logics', to start with. *Ad hoc* modifications of the definition of logic in order to avoid the above mentioned unpleasant examples do not seem to carry much persuasive power —for one thing, a good question is: Where will they stop?

Imagine the following conversation overheard between two philosophers:

(∀belard) 'I bought an arm chair today.'
(∃loise) 'How nice.'
(∀belard) 'It has flatulence filter seat cushion.'
(∃loise) 'Good.'
(∀belard) 'It has a purple upholstered back.'
(∃loise) 'Hmmm...'
(∀belard) 'It has 42 slender chippendale legs.'
(∃loise) 'Wait a moment. I wouldn't call a 'chair' any object having more than 4 legs!'

Now, was ∀belard wrong in using the word 'chair' from the very start? Maybe ∃loise has a sound intuition, and this anomalous object will turn out to be impractical as a chair —its many legs are too difficult to clean, the ensemble is too heavy to carry, or something. Suppose the philosophers will some day agree about the essential properties of a chair, including its maximal number of legs. Will post-modernist designers still have a job? If they will, then what will be the next development to *trivialize* the notion of 'chair'?

Going back to logic, consider the *minimal* tarskian logic defined over some fixed set of formulas $\mathcal{S}$. This logic is characterized by:

$$\Gamma \models^{\mathsf{m}} \Delta \ \text{ iff } \ \Gamma \cap \Delta \neq \varnothing,$$

where $\Gamma \cup \Delta \subseteq \mathcal{S}$. Clearly, this is the minimal logic respecting (C1), and it is easy to check that both (C2) and (C3) are also respected. ∃loise, again, finds this construction quite 'trivial' and dull. Should we then add a further restriction to the definition of logic so as to please her?

It does seem hopeless, and even counterproductive, to expect logicians to reach a final agreement about the answers to fundamental questions such as 'what is logic?', 'what is negation?' (or conjunction, or some other connective), 'what is paraconsistency?' and so on. This does not mean, however, that 'anything goes'. It often seems more realistic and reasonable to look for properties that we do *not* want to allow 'interesting' logics, negations, conjunctions, paraconsistent logics etc to have. This principle that combines a strong wish both for economy and for significance was made transparent as a sort of motto for paraconsistency since its infancy (cf. [5]): 'From the syntactico-semantical standpoint, every mathematical theory is admissible, unless it is trivial' (notice however that the author did not clear up what 'theory' or 'trivial' were supposed to mean). Investing on that idea, clarifying and updating it, the paper [10] shows one way of implementing this *negative* approach to general abstract nonsense. For the purposes of the present paper, it will be sufficient to require non-overcompleteness for the definition of a so-called *minimally decent logic*. From an abstract viewpoint, that can be done by saying for instance that a minimally decent tarskian logic should also respect a further negative axiom, denying the very possibility of semitriviality. From a semantical viewpoint the thing gets a bit more complicated. It is not enough for theorists of valuations to add the requirement that both the set of designated values and the set of undesignated values should be non-empty. One needs also to directly constrain the set of all valuations of an intended semantics —or else collections of dadaistic and nihilistic models might reappear. There is no need to go into details of that here. At any rate, other necessary conditions for minimal decency might of course still impose themselves at some future moment, according to the interest and experience of logic-designers.

Now, at least two lessons may be drawn from the paradox explored in the previous sections. The first lesson is about the usefulness of a multiple-conclusion environment when doing logic in general, and paraconsistent logic in particular (I recommend again checking [10], where this framework was extensively used for the study of *negation*, its more usual positive properties and some negative properties that make it 'minimally decent'). Obviously, as any other formalism, multiple-conclusion will also have its limitations, and the adequacy of its use will depend on the phenomenon that needs to be seized at the time. On the positive side, however, there are several arguments pro multiple-conclusion. Many of them are well-known, or quite obvious, and I will not try to survey them here (for the interested reader, it might be a good idea to check [15]).[4] I will mention only one further particularly interesting advantage of that formalism, as connected to paraconsistency.

Even after the wide acknowledgment of the inferential character of logic, philosophy continues to suffer from a certain 'bias towards truth'. Arguably, because a compact tarskian logic sees no difference between inferences with a finite or an infinite set of premises, because the single-conclusion notation derived from the notion of a closure operator cannot mark the difference between constructive and non-constructive sets of theses, and because of persisting positivistic influences, the logico-philosophical community ended up accommodating with a lot of inertia

[4] A particularly attractive advantage of the multiple-conclusion framework, from a semantical viewpoint, is the so-called *categoricity* of the class of models of each given tarskian logic (check [8]): Any two adequate 2-valued canonical semantics for a given logic $\mathcal{L}$ must be isomorphic. The result is always true for multiple-conclusion logics, and never true for non-overcomplete single-conclusion logics. That explains why each variety of overcompleteness is clearly distinguished from the other varieties in a multiple-conclusion framework, and why the so-called Paradox of Ineffable Inconsistencies is immediately disclosed when one moves from single- to multiple-conclusion consequence relations.

around the notion of theoremhood, as opposed to the notion of inference from a set of premises. Even nowadays, the study of 'logics as sets of theorems' or 'logics as sets of truths' is very likely to find more practitioners than the more inferential-related approach. Besides, even proposals as interesting as those of Łukasiewicz in axiomatizing his modal many-valued logics using the notion of rejected propositions, alongside with accepted propositions, were soon to fall into almost complete disregard (the papers [18, 17, 16] are among the few interesting contemporary exceptions to that trend). But why should truth be privileged over falsehood? Why should acceptance be privileged over rejection?

The multiple-conclusion approach allows not only the inferential character of logic to be taken into proper account but its full symmetry also allows truth and falsehood to be put on equal footing. Playing with the right-left symmetry of the consequence relation turnstile symbol one can very naturally talk for instance about the notion of *duality* of logics, of connectives, and of rules. Given a consequence relation $\rhd$, its dual $\blacktriangleright$ is such that

$$(\Gamma \blacktriangleright \Delta) \text{ iff } (\Delta \rhd \Gamma).$$

Similarly, given any rule of a connective in the first consequence relation, $\rhd$, one can immediately look for the corresponding rule of the dual connective in the second consequence relation, $\blacktriangleright$, just reading the rule the other way around. This way an introduction rule for classical conjunction can be characterized as dual to an elimination rule for classical disjunction, implication can be characterized as dual to right residuation, negation as consistency (explosion) as dual to negation as completeness or determinedness (excluded middle). Any definition involving paraconsistency can immediately be converted into a definition involving its dual, paracompleteness. In semantical terms, given a two-valued interpretation of a tarskian logic, its dual is obtained by uniformly substituting 'true' for 'false', and vice-versa.

It is about time for the 'single-conclusion bias' to be defeated once and for all. If not just for the hidden prejudice against multiple-conclusion, or plain sluggishness of many logicians, the only extra reason I see for no version of the above paradox to have been explicitly reported before (as far as I know) is because there seems not to have been much interest in exploring single-premise inferences (that idea has been taken forward, though, in papers such as [20]). Notice, at any rate, that at the single-premise-single-conclusion case the interpretation of the entailment sign, $\models$, confuses itself with the interpretation of the (classical) material conditional.

If you recall the definition of a $\sim$-consistent logic proposed in the last section, you will see that an $\sim$-inconsistent logic will either be overcomplete or it will disrespect *pseudo-scotus*. Once a *paraconsistent logic* is A LOGIC before anything else, it should be a minimally decent logic, hence it should be $\sim$-inconsistent but not overcomplete (notice that the failure of *pseudo-scotus* is perfectly compatible with dadaism). It is sad to recognize that, several decades after its initial developments, paraconsistent logic remains by and large a terrain wide open for adventurers and for intellectual impostures. The general inability demonstrated by the paraconsistent community so far in having constructive conversations attests to the great lack of coordination in the field. These last grumpy (yet justified) comments of mine might help explaining, at least partially, the serious lack of foundational papers which would help in finally setting some necessary conditions for minimal decency in paraconsistent logic. The second lesson of the present paper intends to be a contribution to that. Instead of proposing changes to the very definition of paraconsistency —say, to those hazy definitions recorded in section 1— my sole suggestion here is that a *minimally decent paraconsistent logic* should, in a multiple-conclusion abstract environment, avoid not only *pseudo-scotus* but also *ex contradictione* —as it has generally been done in the single-conclusion environment, where the two rules are indistinguishable. Semantically, as observed in section 2, this amounts to requiring the semantics of minimally decent (tarskian) paraconsistent logics not just to allow for $\sim$-inconsistent models, but, more specifically, to allow for non-dadaistic $\sim$-inconsistent models.

ACKNOWLEDGEMENTS

This study was partially supported by FCT (Portugal) and FEDER (European Union), namely, via the Project FibLog POCTI / MAT / 37239 / 2001 of the Center for Logic and Computation (IST, Portugal) and the FCT grant SFRH / BD / 8825 / 2002. This research was developed also under the scope of the cooperation agreement 24P-22335/04 between CLE / UNICAMP and CLC / UTL. I take the chance to thank Carlos Caleiro, Walter Carnielli and two anonymous referees for their many helpful comments on an earlier version of this paper.

João Marcos
http://geocities.com/jm_logica/
Department of Informatics and Applied Mathematics, CCET / UFRN, Brazil
Center for Logic and Computation, IST / UTL, Portugal

BIBLIOGRAPHY

[1] J.-Y. Béziau, editor. *Logica Universalis*. Birkhäuser Verlag, Basel, Switzerland, 2005.
[2] Jean-Yves Béziau. Universal Logic. In T. Childers and O. Majers, editors, Logica'94*, Proceedings of the* VIII *International Symposium*, pages 73–93. Czech Academy of Science, Prague, CZ, 1994.
[3] Jean-Yves Béziau. Non-truth-functional many-valuedness. In J.-Y. Béziau, A. Costa-Leite, and A. Facchini, editors, *Aspects of Universal Logic*, pages 199–218. Centre de Recherches Sémiologiques, Neuchâtel, 2004.
[4] Walter A. Carnielli, Marcelo E. Coniglio, and João Marcos. Logics of Formal Inconsistency. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 14. Kluwer Academic Publishers, 2nd edition, 2006. In print.
[5] Newton C. A. da Costa. Observações sobre o conceito de existência em matemática. *Anuário da Sociedade Paranaense de Matemática*, 2:16–19, 1959.
[6] Newton C. A. da Costa and Jean-Yves Béziau. La théorie de la valuation en question. In M. Abad, editor, *Proceedings of the* IX *Latin American Symposium on Mathematical Logic, Part 2* (Bahía Blanca, 1992), pages 95–104. Bahía Blanca: Universidad Nacional del Sur, 1994.

[7] Newton C. A. da Costa and Jean-Yves Béziau. Théorie de la valuation. *Logique et Analyse (N.S.)*, 37(146):95–117, 1994.
[8] Gary M. Hardegree. Completeness and super-valuations. *Journal of Philosophical Logic*, 34(1):81–95, 2005.
[9] Andrea Loparić and Newton C. A. da Costa. Paraconsistency, paracompleteness, and valuations. *Logique et Analyse (N.S.)*, 27(106):119–131, 1984.
[10] João Marcos. On negation: Pure local rules. *Journal of Applied Logic*, 3(1):185–219, 2005. Preprint available at:
`http://www.cle.unicamp.br/e-prints/revised-version-vol_4,n_4,2004.html`.
[11] Dana Scott. On engendering an illusion of understanding. *Journal of Philosophy*, 68:787–807, 1971.
[12] Dana Scott. Completeness and axiomatizability in many-valued logic. In *Proceedings of the Tarski Symposium* (Proc. Sympos. Pure Math., Vol. XXV, held at UC Berkeley, 1971), pages 411–435, Providence / RI, 1974. American Mathematical Society.
[13] Dana Scott. Rules and derived rules. In S. Stenlund, editor, *Logical Theory and Semantical Analysis*, pages 147–161. D. Reidel, Dordrecht, 1974.
[14] D. J. Shoesmith and Timothy J. Smiley. Deducibility and many-valuedness. *The Journal of Symbolic Logic*, 36(4):610–622, 1971.
[15] D. J. Shoesmith and Timothy J. Smiley. *Multiple-Conclusion Logic*. Cambridge University Press, Cambridge / MA, 1978.
[16] Tomasz Skura. Aspects of refutation procedures in the intuitionistic logic and related modal systems. *Logika*, 20, 2000.
[17] Timothy Smiley. Rejection. *Analysis (Oxford)*, 56(1):1–9, 1996.
[18] Allard M. Tamminga. Logics of rejection: Two systems of natural deduction. *Logique et Analyse (N.S.)*, 146:169–208, 1994.
[19] Alfred Tarski. On the concept of following logically. *History and Philosophy of Logic*, 23:155–196, 2002. Translated by M. Stroińska and D. Hitchcock. Original versions published in Polish under the title 'O pojciu wynikania logicznego', in *Przegląd Filozoficzny*, 39:58–68, 1936, and then in German under the title 'Über den Begriff der logischen Folgerung', in *Actes du Congrès International de Philosophie Scientifique*, 7:1–11, 1936.
[20] Igor Urbas. Dual-intuitionistic logic. *Notre Dame Journal of Formal Logic*, 37(3):440–451, 1996.
[21] Ryszard Wójcicki. *Theory of Logical Calculi*. Kluwer, Dordrecht, 1988.
[22] Jan Zygmunt. *An Essay in Matrix Semantics for Consequence Relations*. Wydawnictwo Uniwersytetu Wrocławskiego, Wrocław, 1984.