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LOGICS OF FORMAL INCONSISTENCY

1 INTRODUCTION

1.1 Contradictoriness and inconsistency, consistency and non-contradictoriness

In traditional logic, contradictoriness (the presence of contradictions in a theory or in a body of knowledge) and triviality (the fact that such a theory entails all possible consequences) are assumed inseparable, granted that negation is available. This is an effect of an ordinary logical feature known as 'explosiveness': According to it, from a contradiction 'α and $\neg\alpha$' everything is derivable. Indeed, classical logic (and many other logics) equate 'consistency' with 'freedom from contradictions'. Such logics forcibly fail to distinguish, thus, between contradictoriness and other forms of inconsistency. Paraconsistent logics are precisely the logics for which this assumption is challenged, by the rejection of the classical 'consistency presupposition'. The *Logics of Formal Inconsistency*, **LFI**s, object of this chapter, are the paraconsistent logics that neatly balance the equation:

CONTRADICTIONS + CONSISTENCY = TRIVIALITY

The **LFI**s have a remarkable way of reintroducing consistency into the non-classical picture: They internalize the very notions of consistency and inconsistency at the object-language level. The result of that strategy is the design of very expressive logical systems, whose fundamental feature is the ability of recovering all consistent reasoning right on demand, while still allowing for some inconsistency to linger, otherwise.

Paraconsistency is the study of contradictory yet non-trivial theories.[1] The significance of paraconsistency as a philosophical program which dares to go beyond consistency lies in the possibilities (formal, epistemological and mathematical) to take profit from the distinctions and contrasts between asserting opposites (either in a formal or in a natural language) and ensuring non-triviality (in a theory, formal or not). A previous entry [Priest, 2002] in this Handbook was dedicated to paraconsistent logics. Although partaking in the same basic views on paraconsistency, our approach is oriented towards investigating and exhibiting the features of an ample and very expressive class of paraconsistent logics —the above mentioned **LFI**s.

[1]Paraconsistency has the meaning of 'besides, beyond consistency', as paradox means 'besides, beyond opinion' and 'paraphrase' means 'to phrase in other words'.

D. Gabbay and F. Guenthner (eds.),
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Moreover, our chapter starts from clear-cut abstract definitions of the terms involved (triviality, consistency, paraconsistency, etc.) and analyzes both proof-theoretical and model-theoretical aspects of **LFI**s, insisting on their special interest and hinting about their near ubiquity in the paraconsistent realm.

Once inconsistency is locally allowed, the chief value of a useful logical system (understood as a derivability formalism reflecting some given theoretical or pragmatical constraints) turns out to be its capability of doing what it is supposed to do, namely, to set acceptable inferences apart from unacceptable ones. The least one would ask for is, thus, that the system *does* separate propositions (into two non-empty classes, the derivable ones and the non-derivable) or, in other words, that it be non-trivial. Therefore, the most fundamental guiding criterion for choosing theories and systems worthy of investigation, as suggested by [Jaśkowski, 1948], [Nelson, 1959] and [da Costa, 1959], and extended in [Marcos, 2004b], should indeed be their abstract character of non-triviality, rather than the mere absence of contradictions.

The big challenge for paraconsistentists is to avoid allowing contradictory theories to explode and derive anything else (as they do in classical logic) and *still* to reserve resources to design a respectful logic. For that purpose they must weaken their logical machinery by abandoning explosion in order to be able to draw reasonable conclusions from those theories, and *yet* come up with a legitimate logical system. A current trend in logic has been that of internalizing metatheoretical notions and devices at the object-language level, in order to build ever more expressive logical systems, as in the case of labelled deductive systems, hybrid logics, or the logics of provability. The **LFI**s constitute exactly the class of paraconsistent logics which can internalize the metatheoretical notions of consistency and inconsistency. As a consequence, despite constituting fragments of consistent logics, the **LFI**s can canonically be used to faithfully encode all consistent inferences. We will in this chapter present and discuss these logics, illustrating their uses, properties and representations.

Most of the material for the chapter is based on the article [Carnielli and Marcos, 2002], which founds the formal distinctions between contradictoriness, inconsistency and triviality, which we here utilize. In some cases we correct here the proofs presented there. The **LFI**s, central topic of the present chapter, are carefully defined in Subsection 3.1. All necessary concepts and definitions showing how we approach the property of explosion and how this reflects on the principles of logic will be found in Section 2. Subsection 1.2 serves as vestibular to the more technical sections that follow.

The main **LFI**s are presented in Sections 3 and 4. One of their primary subclasses, the **C**-systems, are introduced as those **LFI**s where consistency can be expressed as a unary connective. Moreover, the **dC**-systems are introduced as those **C**-systems in which the consistency connective may be explicitly defined in terms of other more usual connectives. In Section 3 we study in detail a fundamental example of **LFI**, the logic **mbC**, where consistency is rendered expressible by means of a primitive new connective. This logic is compared to the stronger logic C_1 (cf. [da Costa, 1993] and [da Costa, 1974]), a logic of the early paraconsistent vintage. We provide Hilbert-style axiomatizations, as well as valuation semantics, and adequate tableau systems for **mbC** and C_1. Additionally, adequate possible-translations semantics are proposed for **mbC**.

LFIs are typically based on previously given consistent logics. The fundamental feature enjoyed by classical-based **LFI**s of being able to recover classical reasoning (despite of constituting themselves subsystems of classical logic) is explained in Subsection 3.6.

In Section 4 we extend the logic **mbC** by adding further axioms which permit us to talk about inconsistency and consistency in more symmetric guises inside the logic. A brief study of the thereby obtained logics follows, extending the results obtained in Section 3.

Section 5 explores additional topics on **LFI**s. In Subsection 5.1 some fundamental **dC**-systems are studied. Particular cases of **dC**-systems are da Costa's logics C_n, $1 \leq n < \omega$, Jaśkowski's logic **D2**, and most normal modal logics under convenient formulations. Conveniently extending the previously obtained **LFI**s it is possible to introduce a large family of such logics by controlling the propagation of consistency (cf. Subsection 5.2). This procedure adds flexibility to the game, allowing one to define tailor-suited **LFI**s; we illustrate the case by defining literally thousands of logics, including an interesting class of maximal logics in Subsection 5.3. We end this subsection by a brief note on the possibilities of algebraizing **LFI**s, in general, concluding a series of similar notes and results to be found along the paper, dedicated especially to the difficulties surrounding the so-called replacement property, that guarantees equivalent formulas to be logically indistinguishable.

Section 6 examines some perspectives on the research about Logics of Formal Inconsistency. The chapter ends by a list of axioms and systems given in Section 7.

It goes without saying that the route we will follow in this chapter corresponds not only to our preferences on how to deal with paraconsistency, but brings also a personal choice of topics we consider to be of special philosophical and mathematical relevance.

1.2 The import of the Logics of Formal Inconsistency

Should the presence of contradictions make it impossible to derive anything sensible from a theory or a logic where such contradictions appear, as the classical logician would maintain? Or maybe there are situations in which contradictions are at least temporarily admissible, if only their wild behavior can be somehow controlled? The relevance of such questions show paraconsistency to be a bold program in the foundations of formal sciences. As time goes by, the problems and methods of formal logic, traditionally connected to mathematics and philosophy, can more and more be seen to affect and influence several other areas of knowledge, such as computer science, information systems, formal philosophy, theoretical linguistics, and so forth. In such areas, certainly more than in mathematics, contradictions are presumably unavoidable: If contradictory theories appear only by mistake, or are due to some kind of resource-boundedness on computers, or depend on an altered state of reality, contradictions can hardly be prevented from at least being taken into consideration, as they often show up as gatecrashers. The pragmatic point thus is not whether contradictory theories *exist*, but *how to deal with them*.

Regardless of the disputable status of contradictory theories, it is hard to deny that they are, in many cases, quite *informative*, it being desirable to establish *well-reasoned* judgements even when contradictions are present. Consider, for instance, the following situation (adapted from [Carnielli and Marcos, 2001a]) in which you ask a yes-no question to two people: 'Does Jeca Tatu live in São Paulo?' Exactly one of the three following different scenarios is possible: They might both say 'yes', they might both say 'no', or else one of them might say 'yes' while the other says 'no'. Now, it happens that in no situation you can be sure whether Jeca Tatu lives in São Paulo or not (unless you trust one of the interviewees more than the other), but only in the last scenario, where a contradiction appears, you are sure to have received wrong information from one of your sources.

A challenge of any study on paraconsistency is to oppugn the tacit assumption that contradictory theories necessarily contain false sentences. Thus, if we can build models of structures in which some (but not all) contradictory sentences are simultaneously true, we will have the possibility of maintaining contradictory sentences inside a given theory and still be able, in principle, to perform reasonable inferences from that theory. The problem will not be that of *validating falsities*, but rather of *extending our notion of truth* (an idea further explored, for instance, in [Bueno, 1999]).

In the first half of the last century, some authors, including Łukasiewicz and Vasiliev, proposed a new approach to the idea of non-contradiction, offering interpretations to formal systems in which contradictions could make sense. Between the 1940s and the 60s the first systems of paraconsistent logic appeared (cf. [Jaśkowski, 1948], [Nelson, 1959], and [da Costa, 1993]). For historical notes on paraconsistency we suggest [Arruda, 1980], [D'Ottaviano, 1990], [da Costa and Marconi, 1989], the references mentioned in part 1 of [Priest *et al.*, 1989] and in section 3 of [Priest, 2002], as well as the book [Bobenrieth-Miserda, 1996].

Probably around the 40s, time was ripe for thinking about the role of negation in different terms: The falsificationism of K. Popper (cf. [Popper, 1959]) supported the idea (and stressed its role in the philosophy of science) that falsifying a proposition, in the sense of refuting it, is not the same as assuming the sentence to be false. This apparently led Popper to think about a paraconsistent-like logic dual to intuitionism in his [Popper, 1948], later to be rejected as 'too weak as to be useful' (cf. [Popper, 1989]). But it should be remarked that Popper never dismissed this kind of approach as nonsensical. His disciple D. Miller in [Miller, 2000] in fact argues that the logic for dealing with unfalsifiedness should be paraconsistent.[2] Another proposal by Y. Shramko also defends the paraconsistent character of falsificationism (cf. [Shramko, 2004]).

When proposing his first paraconsistent logics (cf. [da Costa, 1993]) da Costa's idea was that the 'consistency' (which he dubbed 'good-behavior') of a given formula, would not only be a sufficient requisite to guarantee its explosive character, but could also be represented as an ordinary formula of the underlying language. For his initial logic, C_1, he chose to represent the consistency of a formula α by the formula $\neg(\alpha \wedge \neg\alpha)$, and referred to this last formula as a realization of the 'Principle of Non-Contradiction'.

In the present approach, as in [Carnielli and Marcos, 2002], we introduce consistency as a *primitive notion* of our logics: The Logics of Formal Inconsistency, **LFI**s, are paraconsistent logics that internalize the notions of consistency and inconsistency at the object-language level. In this chapter we will also study some significative subclasses of **LFI**s, the **C**-systems and **dC**-systems based on classical logic (of which the logics C_n will only appear as very particular examples).

It is worth noting that, in general, paraconsistent logics do not validate contradictions nor, equivalently, invalidate the 'Principle of Non-Contradiction', in our reading of it (cf. the principle (1) in Subsection 2.1). Most paraconsistent logics, in fact, are proper fragments of (some version of) classical logic, and thus they cannot be contradictory.

Clearly, the concept of paraconsistency is related to the properties of a negation inside a given logic. In that respect, arguments can be found in the literature to the effect that 'negations' of paraconsistent logics would not be proper negation operators (cf. [Slater, 1995] and [Béziau, 2002a]). Béziau's argument amounts to a request for the definition of some minimal 'positive properties' in order to characterize paraconsistent negation as constituting a real *negation* operator, instead of something else. Slater argues for the *inexistence* of paraconsistent logics, given that their negation operator is not a 'contradictory-forming functor', but just a 'subcontrary-forming one', revisiting and extending an earlier argument from [Priest and Routley, 1989]. A reply to this kind of criticism is that it is as convincing as arguing that a 'line' in hyperbolic geometry is not a real line, since, through a given

[2]Indeed, Miller even proposes that the logic C_1 of da Costa's hierarchy could be used as a logic of falsification.

point not on the line, the 'parallel-forming functor' does not define a unique line.[3] In any case, this is not the only possible counter-objection, and the development of paraconsistent logic is not deterred by this discussion. Investigations about the general properties of paraconsistent negations include [Avron, 2002], [Béziau, 1994] and [Lenzen, 1998], among others. Those studies are surveyed in [Marcos, 2004b], where also a minimal set of 'negative properties' for negation is advanced as a new starting point for a unifying study of negation.

2 WHY'S AND HOW'S: CONCEPTS AND DEFINITIONS

2.1 The principles of logic revisited

Our presentation in what follows is situated at the level of a general theory of consequence relations. Let $\wp(X)$ be the powerset of a set X. As usual, given a set *For* of formulas, we say that $\Vdash\subseteq\wp(For)\times For$ defines a (*Tarskian*) *consequence relation* on *For* if the following clauses hold, for any formulas α and β, and subsets Γ and Δ of For (formulas and commas at the left-hand side of $\Vdash$ denote, as usual, sets and unions of sets of formulas):

(Con1) $\alpha\in\Gamma$ implies $\Gamma\Vdash\alpha$ (reflexivity)
(Con2) $(\Delta\Vdash\alpha$ and $\Delta\subseteq\Gamma)$ implies $\Gamma\Vdash\alpha$ (monotonicity)
(Con3) $(\Delta\Vdash\alpha$ and $\Gamma,\alpha\Vdash\beta)$ implies $\Delta,\Gamma\Vdash\beta$ (cut)

So, a (*Tarskian*) *logic* **L** will here be defined simply as a structure of the form $\langle For,\Vdash\rangle$, containing a set of formulas and a consequence relation defined on this set. An additional useful property of a logic is compactness, defined as:

(Con4) $\Gamma\Vdash\alpha$ implies $\Gamma^{\mathsf{fin}}\Vdash\alpha$, for some finite $\Gamma^{\mathsf{fin}}\subseteq\Gamma$ (compactness)

We will assume that the language of every logic **L** is defined over a propositional signature $\Sigma=\{\Sigma_n\}_{n\in\omega}$ such that Σ_n is the set of connectives of arity n. We will also assume that $\mathcal{P}=\{p_n : n\in\omega\}$ is the set of propositional variables (or atomic formulas) from which we freely generate the algebra For of formulas using Σ. Another usual property of a logic is structurality. Let ε be an endomorphism in For, that is, ε is the unique homomorphic extension of a mapping from $\mathcal{P}$ into For. A logic is structural if its consequence relation preserves endomorphisms:

(Con5) $\Gamma\Vdash\alpha$ implies $\varepsilon(\Gamma)\Vdash\varepsilon(\alpha)$ (structurality)

In syntactical terms, structurality corresponds to the rule of uniform substitution or the use of schematic axioms and rules.

[3]In hyperbolic geometry the following property, known as the Hyperbolic Postulate, holds: For every line l and point p not on l, there exist at least two distinct lines parallel to l that pass through p.

Any set $\Gamma \subseteq For$ is called a *theory* of $\mathbf{L}$. A theory Γ is said to be *proper* if $\Gamma \neq For$, and a theory Γ is said to be *closed* if it contains all of its consequences: $\Gamma \Vdash \alpha$ iff $\alpha \in \Gamma$, for every formula α. If $\Gamma \Vdash \alpha$ for all Γ, we will say that α is a *thesis* (of this logic).

Unless explicitly stated to the contrary, we will from now on be working with some fixed arbitrary logic $\mathbf{L} = \langle For, \Vdash \rangle$ where For is written in a signature containing a unary 'negation' connective $\neg$ and $\Vdash$ satisfies (Con1)–(Con3) and (Con5).

Let Γ be a theory of $\mathbf{L}$. We say that Γ is *contradictory with respect to* $\neg$, or simply *contradictory*, if it satisfies:

$$\exists \alpha (\Gamma \Vdash \alpha \text{ and } \Gamma \Vdash \neg\alpha)$$

(The formal framework to deal with this kind of metaproperties can be found in [Coniglio and Carnielli, 2002].) For any such formula α we may also say that Γ is *α-contradictory*.

A theory Γ is said to be *trivial* if it satisfies:

$$\forall \alpha (\Gamma \Vdash \alpha)$$

Of course the theory For is trivial, given (Con1). We can immediately conclude that contradictoriness is a necessary (but, in general, not sufficient) condition for triviality in a given theory, since a trivial theory derives everything.

A theory Γ is said to be *explosive* if:

$$\forall \alpha \forall \beta (\Gamma, \alpha, \neg\alpha \Vdash \beta)$$

Thus, a theory is called explosive if it trivializes when exposed to a pair of contradictory formulas. Evidently, if a theory is trivial, then it is explosive by (Con2). Also, if a theory is contradictory and explosive, then it is trivial by (Con3).

The above definitions can be immediately upgraded from theories to logics. We will say that $\mathbf{L}$ is *contradictory* if all of its theories are contradictory, that is:

$$\forall \Gamma \exists \alpha (\Gamma \Vdash \alpha \text{ and } \Gamma \Vdash \neg\alpha)$$

In the same spirit, we will say that $\mathbf{L}$ is *trivial* if all of its theories are trivial, and $\mathbf{L}$ is *explosive* if all of its theories are explosive.

Because of the monotonicity property (Con2), it is clear that a Tarskian logic $\mathbf{L}$ is contradictory / trivial / explosive if, and only if, its empty theory is contradictory / trivial / explosive.

We are now in position to give a formal definition for some *logical principles* as applied to a generic logic $\mathbf{L}$:

Principle of Non-Contradiction

$$\exists \Gamma \forall \alpha (\Gamma \nVdash \alpha \text{ or } \Gamma \nVdash \neg\alpha) (\mathbf{L} \text{ is non-contradictory}) \quad (1)$$

Principle of Non-Triviality

$$\exists\Gamma\exists\alpha(\Gamma \nVdash \alpha)(\mathbf{L} \text{ is non-trivial}) \tag{2}$$

Principle of Explosion

$$\forall\Gamma\forall\alpha\forall\beta(\Gamma, \alpha, \neg\alpha \Vdash \beta)(\mathbf{L} \text{ is explosive}) \tag{3}$$

The last principle is also often referred to as *Pseudo-Scotus* or Principle of *Ex Contradictione Sequitur Quodlibet*.[4]

It is clear that the three principles are interrelated:

THEOREM 1.
(i) A trivial logic is both contradictory and explosive.
(ii) A logic in which the Principle of Explosion holds fails the Principle of Non-Triviality if, and only if, it fails the Principle of Non-Contradiction. ■

The logics disrespecting (1) are sometimes called *dialectical*. The great majority of the paraconsistent logics in the literature (including the ones studied here) are *not* dialectical. They usually have non-contradictory empty theories, and thus their axioms are non-contradictory, and their inference rules do not generate contradictions from these axioms. All paraconsistent logics which we will present here are in some sense more careful than classical logic, once they extract less consequences than classical logic extracts from the same given theory, or at most the same set of consequences, but never more. The paraconsistent logics studied in the present chapter (as most paraconsistent logics in the literature) do not validate any bizarre form of reasoning, and do not beget contradictory consequences, if such consequences were already not derived in classical logic.

2.2 Paraconsistency: Between explosion and triviality

As mentioned before, some decades ago, Stanisław Jaśkowski ([Jaśkowski, 1948]), David Nelson ([Nelson, 1959]), and Newton da Costa ([da Costa, 1993]), the founders of paraconsistent logic, proposed, independently, the study of logics which could accommodate contradictory yet non-trivial theories. For da Costa, a logic is *paraconsistent*[5] *with respect to* $\neg$ if it can serve as a basis for $\neg$-contradictory yet non-trivial theories, that is:

$$\exists\Gamma\exists\alpha\exists\beta(\Gamma \Vdash \alpha \text{ and } \Gamma \Vdash \neg\alpha \text{ and } \Gamma \nVdash \beta) \tag{4}$$

[4]In fact, single-conclusion logics as those we work with here cannot see the difference between *Pseudo-Scotus* and *ex contradictione*, but those principles can be sharply distinguished in a multiple-conclusion environment (cf. [Marcos, 2004b]).

[5]This denomination would be coined only in the 70s by the Peruvian philosopher Francisco Miró Quesada.

Notice that, in our present framework, the notion of a paraconsistent logic has, in principle, nothing to do with the rejection of the Principle of Non-Contradiction, as it is commonly held. On the other hand, it is related to the rejection of the Principle of Explosion. Indeed, Jaskowśki defined a $\neg$-paraconsistent logic as a logic in which (3) fails:

$$\exists\Gamma\exists\alpha\exists\beta(\Gamma, \alpha, \neg\alpha \nVdash \beta) \tag{5}$$

Using (Con1) and (Con3) it is easy to prove that (4) and (5) are equivalent ways of defining a paraconsistent logic. Whenever it is clear from the context, we will omit the $\neg$ symbol and refer simply to *paraconsistent logics*.

It is very important to observe that a logic where all contradictions are equivalent cannot be paraconsistent. To understand that it is convenient first to make precise the concept of equivalence between sets of formulas: Γ and Δ are said to be *equivalent* if

$$\forall\alpha \in \Delta(\Gamma \Vdash \alpha) \text{ and } \forall\alpha \in \Gamma(\Delta \Vdash \alpha)$$

In particular, we say that two formulas α and β are *equivalent* if the sets $\{\alpha\}$ and $\{\beta\}$ are equivalent, that is:

$$(\alpha \Vdash \beta) \text{ and } (\beta \Vdash \alpha)$$

We denote these facts by writing, respectively, $\Gamma \dashv\Vdash \Delta$, and $\alpha \dashv\Vdash \beta$. The equivalence between formulas is clearly an equivalence relation, because of (Con1) and (Con3). However, the equivalence between sets is not, in general, an equivalence relation, unless the following property holds in $\mathbf{L}$:

(Con6) $(\forall\beta \in \Delta)(\Gamma \Vdash \beta$ and $\Delta \Vdash \alpha)$ implies $\Gamma \Vdash \alpha$ (cut for sets)

REMARK 2. (i) In logics defined using finite matrices or Hilbert calculi with schematic axioms and rules, (Con5) and (Con6) hold good. This is the case of most logics mentioned in the present paper.

(ii) (Con1) and (Con6) guarantee that $\dashv\Vdash$ define an equivalence relation over sets of formulas.

(iii) The condition (Con3) is a consequence of (Con1), (Con2), (Con6). Hint: Instantiate, in (Con6), Γ as $\Delta \cup \Gamma$, Δ as $\Gamma \cup \{\alpha\}$, and α as β.

(iv) The condition (Con6) is not a consequence of (Con1), (Con2), (Con3). Consider, for instance, the logic $\mathbf{L}_{\mathbb{R}} = \langle \mathbb{R}, \Vdash \rangle$ such that $\mathbb{R}$ is the set of real numbers, and $\Vdash$ is defined as follows:

$\Gamma \Vdash x$ iff $x \in \Gamma$, or $x = \frac{1}{n}$ for some $n \in \mathbb{N}$, $n \geq 1$, or there is a sequence $(x_n)_{n \in \mathbb{N}}$ contained in Γ such that $(x_n)_{n \in \mathbb{N}}$ converges to x.

It is easy to see that $\mathbf{L}_{\mathbb{R}}$ satisfies (Con1), (Con2) and (Con3). But (Con6) is not valid in $\mathbf{L}_{\mathbb{R}}$. Indeed, take $\Gamma = \emptyset$, $\Delta = \{\frac{1}{n} : n \in \mathbb{N}, n \geq 1\}$ and

$\alpha = 0$. Then the antecedent of (Con6) is true: Every element of Δ is a thesis, and Δ contains the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ which converges to 0. However, the consequent of (Con6) is false: 0 is not a thesis of $\mathbf{L}_{\mathbb{R}}$.
Observe, by the way, that in $\mathbf{L}_{\mathbb{R}}$ the relation $\dashv\Vdash$ between sets of formulas is not transitive: Take Δ as above, and consider $\Delta_0 = \{0\}$ and $\Delta_1 = \{1\}$. Then $\Delta_0 \dashv\Vdash \Delta$ and $\Delta \dashv\Vdash \Delta_1$, but it is not the case that $\Delta_0 \dashv\Vdash \Delta_1$, because $\Delta_1 \nVdash 0$. ■

THEOREM 3. Let $\mathbf{L}^{\mathsf{cs}}$ be a Tarskian logic respecting cut for sets. Then, if all contradictions are equivalent in $\mathbf{L}^{\mathsf{cs}}$, $\mathbf{L}^{\mathsf{cs}}$ is not paraconsistent.

Proof. Take an arbitrary set Γ in $\mathbf{L}^{\mathsf{cs}}$ respecting cut for sets. Suppose that all contradictions are equivalent, that is, for arbitrary α and β, $\{\alpha, \neg\alpha\} \dashv\Vdash \{\beta, \neg\beta\}$. Then, using (Con2), $\Gamma \cup \{\alpha, \neg\alpha\}$ is β-contradictory for an arbitrary β, and in particular $\Gamma, \alpha, \neg\alpha \Vdash \beta$. ■

DEFINITION 4. The logic $\mathbf{L}$ is *consistent* if it is both explosive and non-trivial, that is, if $\mathbf{L}$ respects both (3) and (2). ■

Paraconsistent logics are inconsistent, in that they control explosiveness, but they can do so in a variety of ways. Trivial logics are inconsistent, by the above definition. What distinguishes a paraconsistent logic from a trivial logic is that a trivial logic does not disallow any inference: It accepts everything. As a consequence of the above definition of consistency, a third equivalent approach to the notion of paraconsistency could be proposed, parallel to those from definitions (4) and (5):

$$\text{A logic is paraconsistent if it is inconsistent yet non-trivial.} \tag{6}$$

The compatibility of paraconsistency with the existence of some suitable explosive or trivial proper theories makes some paraconsistent logics able to recover classical reasoning, as we will see in Section 3.6. We will from now on introduce some specializations on the above definitions and principles.

A logic $\mathbf{L}$ is said to be *finitely trivializable* when it has finite trivial theories. Evidently, if a logic is explosive, then it is finitely trivializable. Non-explosive logics might be finitely trivializable or not.

A formula ξ in $\mathbf{L}$ is a *bottom particle* if it can, by itself, trivialize the logic, that is:

$$\forall\Gamma\forall\beta(\Gamma, \xi \Vdash \beta)$$

A bottom particle, when it exists, will here be denoted by $\bot$. This notation is unambiguous in the following sense: Any two bottom particles are equivalent. If in a given logic a bottom particle is also a thesis, then the logic is trivial — in which case, of course, all formulas turn out to be bottom particles.

The existence of bottom particles inside a given logic $\mathbf{L}$ is regulated by the following principle:

Principle of *Ex Falso Sequitur Quodlibet*

$$\exists\xi\forall\Gamma\forall\beta(\Gamma, \xi \Vdash \beta)(\mathbf{L} \text{ has a bottom particle}) \tag{7}$$

As it will be seen, the existence of logics that do not obey (3) while still respecting (7) (as all **LFI**s of the present chapter) shows that *ex contradictione* does not need to be identified with *ex falso*, contrary to what is commonly held.

The dual concept of a bottom particle is that of a *top particle*, that is, a formula ζ which follows from every theory:

$$\forall\Gamma(\Gamma \Vdash \zeta)$$

We will denote any fixed such particle, when it exists, by $\top$ (again, this notation is unambiguous). Evidently, given a logic, any of its theses will constitute such a top particle (and logics with no theses, like Kleene's 3-valued logic, have no such particles). It is easy to see that the addition of a top particle to a given theory is pretty innocuous, for in that case $\Gamma, \top \Vdash \alpha$ if and only if $\Gamma \Vdash \alpha$.

Henceforth, a formula φ of $\mathbf{L}$ constructed using all and only the variables $p_0, \ldots, p_n$ will be denoted by $\varphi(p_0, \ldots, p_n)$. This formula will be said to *depend only* on the variables that occur in it. The notation can be generalized to sets, and the result is denoted by $\Gamma(p_0, \ldots, p_n)$. If $\gamma_0, \ldots, \gamma_n$ are formulas then $\varphi(\gamma_0, \ldots, \gamma_n)$ will denote the (simultaneous) substitution of p_i by γ_i in $\varphi(p_0, \ldots, p_n)$ (for $i = 0, \ldots, n$). Given a set of formulas $\Gamma(p_0, \ldots, p_n)$, we will write $\Gamma(\gamma_0, \ldots, \gamma_n)$ with an analogous meaning.

DEFINITION 5. We say that a logic $\mathbf{L}$ has a *supplementing negation* if there is a formula $\varphi(p_0)$ such that:

(a) $\varphi(\alpha)$ is not a bottom particle, for some α;
(b) $\forall\Gamma\forall\alpha\forall\beta(\Gamma, \alpha, \varphi(\alpha) \Vdash \beta)$. ■

Notice that the same logic might have several non-equivalent supplementing negations (see Remark 36).

Consider a logic having a supplementing negation, and denote it by $\sim$. Parallel to the definition of contradictoriness with respect to $\neg$, we might now define a theory Γ to be *contradictory with respect to* $\sim$ if it is such that:

$$\exists\alpha(\Gamma \Vdash \alpha \text{ and } \Gamma \Vdash {\sim}\alpha)$$

Accordingly, a logic $\mathbf{L}$ could be said to be *contradictory with respect to* $\sim$ if all of its theories were contradictory with respect to $\sim$. Obviously, by design, no logic can be $\sim$-paraconsistent, or even $\sim$-contradictory, and a logic that has a supplementing negation must satisfy the Principle of Non-Contradiction with respect to this negation. The main logics studied in this paper are all endowed with supplementing negations. The availability of some specific supplementing negations makes some paraconsistent logics able to emulate classical negation.

Here we may of course introduce yet another version of (3):

Supplementing Principle of Explosion

$$\textbf{L} \text{ has a supplementing negation} \tag{8}$$

Supplementing negations are very common. We will show here some sufficient conditions for their definition. The presence of a convenient implication in our logics is often convenient so as to help explicitly internalizing the definition of new connectives. Say that a logic **L** has a *deductive implication* if there is a formula $\psi(p_0, p_1)$ such that:

(a) $\psi(\alpha, \beta)$ is not a bottom particle, for some choice of α and β;
(b) $\forall\alpha\forall\beta\forall\Gamma(\Gamma \Vdash \psi(\alpha, \beta)$ implies $\Gamma, \alpha \Vdash \beta)$.
(c) $\psi(\alpha, \beta)$ is not a top particle, for some choice of α and β;
(d) $\forall\alpha\forall\beta\forall\Gamma(\Gamma, \alpha \Vdash \beta$ implies $\Gamma \Vdash \psi(\alpha, \beta))$.

Inside the most usual logics, condition (b) is usually guaranteed by the validity of the rule of *modus ponens*, while condition (d) is guaranteed by the so-called 'deduction theorem', when it holds. Obviously, any logic having a deductive implication will be non-trivial, by condition (a).

THEOREM 6. Let **L** be a logic endowed with a bottom particle $\bot$ and a deductive implication $\to$.
(i) Let $\neg$ be some negation symbol, and suppose that it satisfies:
(a) $\Gamma, \neg\alpha \Vdash \alpha \to \bot$;
(b) $\Gamma, \neg\alpha \to \bot \Vdash \alpha$.
Then, this $\neg$ is a supplementing negation.
(ii) Suppose, otherwise, that the following is the case:
(c) $\alpha \to \bot \nVdash \bot$, for some formula α.
Then, a supplementing negation can be defined by setting $\neg\alpha \stackrel{\text{def}}{=} \alpha \to \bot$.

Proof. Item (i). By hypothesis (a) and the properties of the bottom and the implication, $\Gamma, \alpha, \neg\alpha \Vdash \beta$. Now, suppose $\neg\alpha$ defines a bottom particle, for any choice of α. Then, by the deduction theorem, $\Gamma \Vdash \neg\alpha \to \bot$, for an arbitrary Γ. Thus, by (b) and (Con3), $\Gamma \Vdash \alpha$. But this cannot be the case, as **L** is non-trivial.
Item (ii) is a straightforward consequence of the above definitions, and we leave it as an exercise for the reader. ∎

One might also consider the dual of a supplementing negation:

DEFINITION 7. We say that a logic **L** has a *complementing negation* if there is a formula $\psi(p_0)$ such that:

(a) $\psi(\alpha)$ is not a top particle, for some α;
(b) $\forall\Gamma\forall\alpha(\Gamma, \alpha \Vdash \psi(\alpha)$ implies $\Gamma \Vdash \psi(\alpha))$.

We say that **L** has a *classical negation* if it has some (primitive or defined) negation connective that is both supplementing and complementing. As a particular consequence of this definition, it can be easily checked that for any classical negation $\div$ the following equivalence will be derivable: $(\div\div\alpha \dashv\Vdash \alpha)$. ■

Yet some other versions of explosiveness can be considered:

DEFINITION 8. Let **L** be a logic, and let $\sigma(p_0, \ldots, p_n)$ be a formula of **L**.
(i) We say that **L** is *partially explosive with respect to* the formula σ, or *σ-partially explosive*, if:

(a) $\sigma(\alpha_0, \ldots, \alpha_n)$ is not a top particle, for some choice of $\alpha_0, \ldots, \alpha_n$;

(b) $\forall\Gamma\forall\alpha_0 \ldots \forall\alpha_n\forall\alpha(\Gamma, \alpha, \neg\alpha \Vdash \sigma(\alpha_0, \ldots, \alpha_n))$.

(ii) **L** is *boldly paraconsistent* if it is not partially explosive.

(iii) **L** is said to be *controllably explosive in contact with* the formula σ, if:

(a) $\sigma(\alpha_0, \ldots, \alpha_n)$ and $\neg\sigma(\alpha_0, \ldots, \alpha_n)$ are not bottom particles, for some choice of $\alpha_0, \ldots, \alpha_n$;

(b) $\forall\Gamma\forall\alpha_0 \ldots \forall\alpha_n\forall\beta(\Gamma, \sigma(\alpha_0, \ldots, \alpha_n), \neg\sigma(\alpha_0, \ldots, \alpha_n) \Vdash \beta)$. ■

EXAMPLE 9. A well-known example of a logic which is not explosive but is partially explosive, is given by Kolmogorov and Johánsson's Minimal Intuitionistic Logic, *MIL*, obtained by the addition to the positive part of intuitionistic logic (see axioms (Ax1)–(Ax8) in Definition 23 below) of some weak forms of *reductio ad absurdum* (cf. [Johánsson, 1936] and [Kolmogorov, 1967]). In this logic, the intuitionistically valid inference $(\Gamma, \alpha, \neg\alpha \Vdash \beta)$ fails, but $(\Gamma, \alpha, \neg\alpha \Vdash \neg\beta)$ holds good. This means that *MIL* is paraconsistent, but not boldly paraconsistent, as all negated propositions can be inferred from any given contradiction. A class of (obviously non-boldly) paraconsistent logics extending *MIL* is studied in [Odintsov, 2004]. ■

The requirement that a paraconsistent logic should be boldly paraconsistent was championed by [Urbas, 1990]. The class of boldly paraconsistent logics is indeed very natural and pervasive. From now on, we will be making an effort to square our paraconsistent logics into this class (check Theorems 16, 30 and 122).

Most paraconsistent logics studied in this chapter are also controllably explosive (check, in particular, Theorem 76).

Conjunction plays a central role in relating contradictoriness and triviality. A logic **L** is said to be *left-adjunctive* if there is a formula $\psi(p_0, p_1)$

such that:

(a) $\psi(\alpha, \beta)$ is not a bottom particle, for some α and β;
(b) $\forall\alpha\forall\beta\forall\Gamma\forall\gamma(\Gamma, \alpha, \beta \Vdash \gamma$ implies $\Gamma, \psi(\alpha, \beta) \Vdash \gamma)$.

The formula $\psi(\alpha, \beta)$, when it exists, will often be denoted by $(\alpha \wedge \beta)$, and the sign $\wedge$ will be called a *left-adjunctive conjunction* (but it will not necessarily have, of course, all properties of a classical conjunction). Similarly, a logic **L** is said to be *left-disadjunctive* if there is a formula $\varphi(p_0, p_1)$ such that:

(c) $\varphi(p_0, p_1)$ is not a top particle, for some α and β;
(d) $\forall\alpha\forall\beta\forall\Gamma\forall\gamma(\Gamma, \varphi(\alpha, \beta) \Vdash \gamma$ implies $\Gamma, \alpha, \beta \Vdash \gamma)$.

In general, whenever there is no risk of misunderstanding or of misidentification of different things, we might also denote the formula $\varphi(\alpha, \beta)$, when it exists, by $(\alpha \wedge \beta)$, and we will accordingly call $\wedge$ a *left-disadjunctive conjunction*. Of course, a logic can have just one of these conjunctions, or it can have both a left-adjunctive and a left-disadjunctive conjunction without the two of them coinciding. In natural deduction, clause (b) corresponds to conjunction elimination, and clause (d) to conjunction introduction.

It is straightforward to prove the following:

THEOREM 10. Let **L** be a left-adjunctive logic. (i) If **L** is finitely trivializable (in particular, if it has a supplementing negation), then it has a bottom particle. (ii) If **L** respects *ex contradictione*, then it also respects *ex falso*. ■

EXAMPLE 11. The 'pre-discussive' logic J proposed in [Jaśkowski, 1948], in the usual signature of classical logic, is such that:

$$\Gamma \Vdash_J \alpha \text{ iff } \Diamond\Gamma \Vdash_{S5} \Diamond\alpha,$$

where $\Diamond\Gamma = \{\Diamond\gamma : \gamma \in \Gamma\}$, $\Diamond$ denotes the possibility operator, and $\Vdash_{S5}$ denotes the consequence relation defined by the well-known modal logic $S5$. It is easy to see that $(\alpha, \neg\alpha \Vdash_J \beta)$ does not hold in general, though $(\alpha \wedge \neg\alpha) \Vdash_J \beta$ does hold good, for any formulas α and β. This phenomenon can only happen because J is left-adjunctive but not left-disadjunctive. Hence, Theorem 10 still holds for J, but this logic provides an example of a logic respecting the Principle of *Ex Falso Sequitur Quodlibet* (7) but not the Principle of *Ex Contradictione Sequitur Quodlibet* (3). ■

The literature on paraconsistency (cf. section 4.2 of [Priest, 2002]) traditionally calls *non-adjunctive* the logics failing left-disadjunctiveness.

3 LFIS AND THEIR RELATIONSHIP TO CLASSICAL LOGIC

3.1 *Introducing **LFIs** and **C**-systems*

From now on, we will concentrate on logics which are paraconsistent but nevertheless have some special explosive theories, as those discussed in the last section. With the help of such theories some concepts can be studied under a new light — this is the case of the notion of *consistency* (and its opposite, the notion of *inconsistency*), as we shall see. This section will introduce the Logics of Formal Inconsistency as the paraconsistent logics that respect a certain Gentle Principle of Explosion, to be clarified below. By way of motivation, we start with a few helpful definitions and concrete examples.

Given two logics $\mathbf{L1} = \langle For_1, \Vdash_1 \rangle$ and $\mathbf{L2} = \langle For_2, \Vdash_2 \rangle$, we will say that **L2** is a (proper) *linguistic extension* of **L1** if For_1 is a (proper) subset of For_2, and we will say that **L2** is a (proper) *deductive extension* of **L1** if $\Vdash_1$ is a (proper) subset of $\Vdash_2$. Finally, if **L2** is both a proper linguistic extension and a proper deductive extension of **L1**, and if the restriction of **L2**'s consequence relation $\Vdash_2$ to the set For_1 will make it identical to $\Vdash_1$ (that is, if $For_1 \subset For_2$, and for any $\Gamma \cup \{\alpha\} \subseteq For_1$ we have that $\Gamma \Vdash_2 \alpha$ iff $\Gamma \Vdash_1 \alpha$) then we will say that **L2** is a *conservative extension* of **L1**. In any of the above cases we can more generally say that **L2** is an *extension* of **L1**, or that **L1** is a *fragment* of **L2**. These concepts will be used here to compare a number of logics that will be presented. Most paraconsistent logics in the literature, and all of those studied here, are proper deductive fragments of classical logic in a convenient signature.

REMARK 12. From here on, Σ will denote the signature containing the binary connectives $\wedge$, $\vee$, $\rightarrow$, and the unary connective $\neg$, such that $\mathcal{P} = \{p_n : n \in \omega\}$ is the set of atomic formulas. By *For* we will denote the set of formulas freely generated by $\mathcal{P}$ over Σ.

In the same spirit, Σ° will denote the signature obtained by the addition of a new unary connective $\circ$ to the signature Σ, and For° will denote the algebra of formulas for the signature Σ°. ■

EXAMPLE 13. Consider the logic given by the following matrices:

$\wedge$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\rightarrow$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	0
0	1	1	1

	$\neg$
1	0
1/2	1/2
0	1

where both 1 and $\frac{1}{2}$ are designated values. *Pac* is the name under which this logic appeared in [Avron, 1991] (Section 3.2.2), though it had previously appeared, for instance, in [Avron, 1986], under the denomination $RM_3^{\tilde{\supset}}$, and, even before that, in [Batens, 1980], where it was called PI^s. *Pac* is a conservative extension of the logic LP by the addition of a classical implication. LP is an early example of a 3-valued paraconsistent logic with a classical conjunction and a classical disjunction, and it was introduced in [Asenjo, 1966] and investigated in [Priest, 1979].

In Pac, for no formula α it is the case that $\alpha, \neg\alpha \vdash_{Pac} \beta$ for all β. So, Pac is not a controllably explosive logic, therefore it is, in particular, a paraconsistent logic. A classical negation in Pac would be introduced by the matrix:

	$\sim$
1	0
$^1/_2$	0
0	1

It is clear that such a negation is *not* definable in Pac, though, for any truth-function of this logic having only $\frac{1}{2}$'s as input will also have $\frac{1}{2}$ as output. As a consequence, Pac has no bottom particle (and it cannot express the consistency of its formulas, as we shall see below). Being a left-adjunctive logic as well, it is consequently not finitely trivializable. ■

EXAMPLE 14. In adding to Pac either a supplementing negation as above or a bottom particle, we will obtain a well-known conservative extension of it, called $\mathbf{J}_3$, which is still paraconsistent but has some interesting explosive theories: It satisfies, in particular, principles (7) and (8) from the previous subsection. This logic was introduced in [Schütte, 1960] for proof-theoretical reasons and independently investigated in [D'Ottaviano and da Costa, 1970] as a 'possible solution to the problem of Jaśkowski'. It also reappeared quite often in the literature after that, for instance as the logic **CLuNs** in [Batens and De Clercq, 2000]. In the first presentation of $\mathbf{J}_3$, a 'possibility connective' ∇ was introduced instead of the supplementing negation $\sim$. In [Epstein, 2000] it was presented again, this time having also a sort of 'consistency connective' $\circ$ (originally denoted by ©) as primitive. The truth-tables of ∇ and $\circ$ are as follows:

	∇	$\circ$
1	1	1
$^1/_2$	1	0
0	0	1

The expressive and inferential power of this logic was more deeply explored in [Avron, 1999] and in [Carnielli *et al.*, 2000]. The latter paper also explores the possibility of applying this logic to the study of inconsistent databases (for a more technical perspective see [de Amo *et al.*, 2002]), abandoning $\sim$ and ∇ but still retaining $\circ$ as primitive. This logic (renamed **LFI1** in the signature Σ°) has been argued to be appropriate for formalizing the notion of consistency in a convenient way, as discussed below. It is worth noticing that

$\sim\alpha$ and $\nabla\alpha$ can be defined in **LFI1** as $(\neg\alpha\wedge\circ\alpha)$ and $(\alpha\vee\neg\circ\alpha)$, respectively. On the other hand, $\circ\alpha \stackrel{\text{def}}{=} (\neg\nabla\alpha \vee \neg\nabla\neg\alpha)$. A complete axiomatization for **LFI1** is presented in Theorem 119. ■

EXAMPLE 15. Paraconsistency and many-valuedness have often been companions. In [Sette, 1973] the following 3-valued logic, alias $\mathbf{P}^1$, was studied:

$\wedge$	1	$^1/_2$	0
1	1	1	0
$^1/_2$	1	1	0
0	0	0	0

$\vee$	1	$^1/_2$	0
1	1	1	1
$^1/_2$	1	1	1
0	1	1	0

$\to$	1	$^1/_2$	0
1	1	1	0
$^1/_2$	1	1	0
0	1	1	1

	$\neg$
1	0
$^1/_2$	1
0	1

where 1 and $\frac{1}{2}$ are the designated values. The truth-table of the consistency connective $\circ$ as in Example 14 can now be defined via $\circ\alpha \stackrel{\text{def}}{=} \neg\neg\alpha\vee\neg(\alpha\wedge\alpha)$. The logic $\mathbf{P}^1$ has the remarkable property of being controllably explosive in contact with arbitrary non-atomic formulas, that is, the paraconsistent behavior obtains only at the atomic level: $\alpha, \neg\alpha \vDash \beta$, for arbitrary non-atomic α. Moreover, another property of this logic is that $\vDash \circ\alpha$ holds for non-atomic α. Those two properties are in fact not related by a mere accident, but as an instance of Theorem 76. A complete axiomatization for the logic $\mathbf{P}^1$ is presented at Theorem 119. ■

We had committed ourselves to present paraconsistent logics that would be boldly paraconsistent (recall Definition 8). The above logics have indeed this property:

THEOREM 16. **LFI1** and $\mathbf{P}^1$ are boldly paraconsistent. And so are their fragments.

Proof. Assume $\Gamma \nvDash \sigma(p_0, \ldots, p_n)$ for some appropriate choice of formulas. In particular, by (Con2), it follows that $\nvDash \sigma(p_0, \ldots, p_n)$. Now, consider a variable p not in $p_0, \ldots, p_n$. Let p be assigned the value $\frac{1}{2}$, and extend this assignment to the variables $p_0, \ldots, p_n$ so as to give the value 0 to $\sigma(p_0, \ldots, p_n)$. It is obvious that, in this situation, $p, \neg p \nvDash \sigma(p_0, \ldots, p_n)$. ■

Paraconsistent logics are tools for reasoning under conditions which do not presuppose consistency. If we understand consistency as what might be lacking to a contradiction for it to become explosive, logics like **LFI1** and $\mathbf{P}^1$ are clearly able to express the consistency (of a formula) at the object-language level. This feature will permit consistent reasoning to be recovered from inside an inconsistent environment.

In formal terms, consider a (possibly empty) set $\bigcirc(p)$ of formulas which depends only on the propositional variable p, satisfying the following. There are formulas α and β such that:

(a) $\bigcirc(\alpha), \alpha \nVdash \beta$;
(b) $\bigcirc(\alpha), \neg\alpha \nVdash \beta$.

We will call a theory Γ *gently explosive* (*with respect to* $\bigcirc(p)$) if:

$$\forall\alpha\forall\beta(\Gamma, \bigcirc(\alpha), \alpha, \neg\alpha \Vdash \beta).$$

A gently explosive theory Γ will be said to be *finitely* so when $\bigcirc(p)$ is a finite set.

A logic **L** will be said to be *(finitely) gently explosive* when there is a (finite) set $\bigcirc(p)$ such that all of the theories of **L** are (finitely) gently explosive (with respect to $\bigcirc(p)$). Notice that a finitely gently explosive theory is finitely trivialized in a very distinctive way.

We may now consider the following 'gentle' versions of the Principle of Explosion:

Gentle Principle of Explosion

$$\textbf{L} \text{ is gently explosive with respect to some set } \bigcirc(p) \tag{9}$$

Finite Gentle Principle of Explosion

$$\textbf{L} \text{ is gently explosive with respect to some finite set } \bigcirc(p) \tag{10}$$

For any formula α, the set $\bigcirc(\alpha)$ will express, in a specific sense, the consistency of α relative to the logic **L**. When this set is a singleton, we will denote by $\circ\alpha$ the sole element of $\bigcirc(\alpha)$, and in this case $\circ$ defines a *consistency connective* or *consistency operator*.

The above definitions are very natural, and paraconsistent logics with a consistency connective are indeed quite common. To see that, say that a logic **L** *has a classical disjunction* if there is a formula $\psi(p_0, p_1)$ such that:

$$\text{(a) } \psi(\alpha, \beta) \text{ is not a bottom particle, for some } \alpha \text{ and } \beta;$$
$$\text{(b) } \forall\alpha\forall\beta\forall\Gamma\forall\Delta\forall\gamma(\Gamma, \alpha \Vdash \gamma \text{ and } \Delta, \beta \Vdash \gamma \text{ implies } \Gamma, \Delta, \psi(\alpha, \beta) \Vdash \gamma).$$
$$\text{(c) } \psi(\alpha, \beta) \text{ is not a top particle, for some } \alpha \text{ and } \beta;$$
$$\text{(d) } \forall\alpha\forall\beta\forall\Gamma\forall\gamma(\Gamma, \psi(\alpha, \beta) \Vdash \gamma \text{ implies } \Gamma, \alpha \Vdash \gamma \text{ and } \Gamma, \beta \Vdash \gamma).$$

In natural deduction, clause (b) corresponds to disjunction elimination, and clause (d) to disjunction introduction. The reader can now easily check that:

THEOREM 17. (i) Any non-trivial explosive theory / logic is finitely gently explosive, supposing that there is some formula α such that $\neg\alpha$ is not a bottom particle. (ii) Any left-adjunctive finitely gently explosive logic respects *ex falso*. (iii) Let **L** be a logic containing a bottom particle $\bot$, a classical disjunction $\vee$, an implication $\rightarrow$ respecting *modus ponens* and a negation $\neg$ such that there exists some formula α satisfying:
(a) $\alpha, (\neg\alpha \rightarrow \bot) \nVdash \bot$;
(b) $\neg\alpha, (\alpha \rightarrow \bot) \nVdash \bot$.
Then **L** defines a consistency operator $\circ\alpha \stackrel{\text{def}}{=} (\alpha \rightarrow \bot) \vee (\neg\alpha \rightarrow \bot)$. ■

We now define the Logics of Formal Inconsistency as the paraconsistent logics that can 'talk about consistency' in a meaningful way.

DEFINITION 18. A *Logic of Formal Inconsistency* (**LFI**) is any logic in which explosion (3) does not hold, but gentle explosion (9) holds. ■

Besides the 3-valued paraconsistent logics presented in the above examples, we will study in this chapter several other paraconsistent logics based on different kinds of semantics. Many will have been originally proposed without a primitive consistency connective, but, being sufficiently expressive, they will often be shown to admit of such a connective. Examples of that phenomenon were already presented above, for the cases of **LFI1** and $\mathbf{P}^1$. Another interesting and maybe even surprising example of that is given by Jaśkowski's Discussive Logic **D2** (cf. [Jaśkowski, 1948] and [Jaśkowski, 1949]), the first paraconsistent logic ever introduced in the literature.

EXAMPLE 19. Let $\Sigma^{\Diamond}$ be the extension of the signature Σ obtained by the addition of a new unary connective $\Diamond$, and let $For^{\Diamond}$ be the corresponding algebra of formulas. Let $\Vdash_{S5}$ be the consequence relation of modal logic *S5* over the language $For^{\Diamond}$. Consider a mapping $* : For \longrightarrow For^{\Diamond}$ such that:

1. $p^* = p$ for every $p \in \mathcal{P}$;
2. $(\neg\alpha)^* = \neg\alpha^*$;
3. $(\alpha \vee \beta)^* = \alpha^* \vee \beta^*$;
4. $(\alpha \wedge \beta)^* = \alpha^* \wedge \Diamond\beta^*$;
5. $(\alpha \to \beta)^* = \Diamond\alpha^* \to \beta^*$.

Given $\Gamma \subseteq For$, let Γ^* denote the subset $\{\alpha^* \ : \ \alpha \in \Gamma\}$ of $For^{\Diamond}$. For any $\Gamma \subseteq For^{\Diamond}$ let $\Diamond\Gamma = \{\Diamond\alpha \ : \ \alpha \in \Gamma\}$. Jaśkowski's Discussive logic **D2** is defined over the signature Σ as follows: $\Gamma \Vdash_{\mathbf{D2}} \alpha$ iff $\Diamond\Gamma^* \Vdash_{S5} \Diamond\alpha^*$, for any $\Gamma \cup \{\alpha\} \subseteq For$. With this definition, **D2** is not explosive with respect to the negation $\neg$, that is, **D2** is paraconsistent (with respect to $\neg$). Consider now the following abbreviations defined on the set For (here, $\alpha \in For$):

$\top \stackrel{\text{def}}{=} (\alpha \vee \neg\alpha)$;
$\bot \stackrel{\text{def}}{=} \neg\top$;
$\blacksquare\alpha \stackrel{\text{def}}{=} (\neg\alpha \to \bot)$;
$\blacklozenge\alpha \stackrel{\text{def}}{=} \neg\blacksquare\neg\alpha$;
$\circ\alpha \stackrel{\text{def}}{=} (\blacklozenge\alpha \to \blacksquare\alpha)$.

It is possible to show that, in **D2**, the formulas $\top$ and $\bot$ are top and bottom particles, respectively, and $\circ$ is a consistency operator. ■

THEOREM 20.
(i) Classical logic is not an **LFI**.

(ii) *Pac* (see Example 13) is also not an **LFI**.

(iii) **LFI1** (see Example 14) is an **LFI**.

(iv) $\mathbf{P}^1$ (see Example 15) is an **LFI**.

(v) Jaśkowski's Discussive Logic **D2** (see Example 19) is an **LFI**.

Proof. For item (i), note that explosion (3) holds classically.

To check item (ii), let p be an atomic formula and let $\bigcirc(p)$ be the set of all formulas of *Pac* that depend only on p. The matrix valuation that assigns $\frac{1}{2}$ to p and 0 to q is a model for $\bigcirc(p), p, \neg p$ but it invalidates gentle explosion on q.

For item (iii), take consistency to be expressed in $\mathbf{J}_3$ by the connective $\circ$, as intended, that is, take $\bigcirc(\alpha) = \{\circ\alpha\}$. Obviously, $\bigcirc(\alpha), \alpha, \neg\alpha \vDash \beta$ holds. Take now a matrix valuation that assigns 1 to p and notice that $\bigcirc(p), p \nvDash \beta$. Finally, take a valuation that assigns 0 to p and notice that $\bigcirc(p), \neg p \nvDash \beta$.

To check item (iv), again take consistency to be expressed in $\mathbf{P}^1$ by $\circ$ and note that $p, \neg p \nvDash q$, for atomic and distinct p and q.

Item (v) can be checked from the definitions on Example 19. ■

In accordance with definition (6) from Subsection 2.2, paraconsistent logics are the non-trivial logics whose negation fails the 'consistency presupposition'. Some inferences that depend on this presupposition, thus, will necessarily be lost. However, one might well expect that, if a sufficient number of 'consistency assumptions' are made, then those same inferences should be recovered. In fact, the **LFI**s are intended to be exactly the logics that can internalize this idea. To be more precise, and following [Marcos, 2004a]:

REMARK 21. Consider a logic $\mathbf{L}1 = \langle For_1, \Vdash_1 \rangle$ in which explosion holds good for a negation $\neg$, that is, a logic that satisfies, in particular, the rule $(\alpha, \neg\alpha \Vdash_1 \beta)$. Let $\mathbf{L}2 = \langle For_2, \Vdash_2 \rangle$ now be some other logic written in the same signature as $\mathbf{L}1$ such that: (i) $\mathbf{L}2$ is a proper deductive fragment of $\mathbf{L}1$ that only validates inferences of $\mathbf{L}1$ that are compatible with the *failure* of explosion; (ii) $\mathbf{L}2$ is *expressive* enough so as to be an **LFI**, therefore, in particular, there will be in $\mathbf{L}2$ a set of formulas $\bigcirc(p)$ such that $(\bigcirc(\alpha), \alpha, \neg\alpha \Vdash_2 \beta)$ holds good; (iii) $\mathbf{L}1$ can in fact be *recovered* from $\mathbf{L}2$ by the addition of $\bigcirc(\alpha)$ as a new set of valid schemas / axioms. These constraints alone suggest that the reasoning of $\mathbf{L}1$ might somehow be recovered from inside $\mathbf{L}2$, if only a sufficient number of 'consistency assumptions' are added in each case. Thus, typically the following *Derivability Adjustment Theorem* (**DAT**) can be proved:

$$\forall\Gamma\forall\gamma\exists\Delta(\Gamma \Vdash_1 \gamma \text{ iff } \bigcirc(\Delta), \Gamma \Vdash_2 \gamma).$$

The **DAT** shows how the weaker logic **L**2 can be used to 'talk about' the stronger logic **L**1. The essential idea behind such theorem was emphasized in [Batens, 1980], but an early version of that very idea can already be found in [da Costa, 1993] and [da Costa, 1974] (check our Theorem 110). On those grounds, **LFI**s are thus proposed as the non-trivial inconsistent logics that can recover consistent inferences through convenient derivability adjustments. ■

To get a bit more concrete, and at the same time specialize from the broad Definition 18 of **LFI**s, we introduce now the concept of a **C**-system.

Consider a logic $\mathbf{L} = \langle \widehat{For}, \Vdash \rangle$ defined over a signature $\widehat{\Sigma}$ that includes a negation symbol $\neg$. Call $\widehat{For}^+$ the set of all *¬-free formulas* of **L**, that is, the algebra of formulas defined over the signature obtained from Σ eliminating the negation symbol $\neg$.

Consider logics $\mathbf{L1} = \langle \widehat{For}_1, \Vdash_1 \rangle$ and $\mathbf{L2} = \langle \widehat{For}_2, \Vdash_2 \rangle$ defined over a signature $\widehat{\Sigma}$ as above. We say that **L**1 is *¬-less-equivalent to* **L**2 if:

(a) $\widehat{For}_1^+ = \widehat{For}_2^+$;

(b) $(\Gamma \Vdash_1 \alpha \Leftrightarrow \Gamma \Vdash_2 \alpha)$, for all $\Gamma \cup \{\alpha\} \subseteq \widehat{For}_1^+$.

Of course, this is an equivalence relation.

DEFINITION 22. A logic **L**2 is said to be a **C**-*system based on* **L**1 (in short, a **C**-system) if:

(a) **L**2 is an **LFI** such that the set $\bigcirc(p)$ is a singleton,

(b) **L**1 is not ¬-paraconsistent, and

(c) **L**2 is ¬-less-equivalent to **L**1. ■

All **C**-systems we will be studying below are non-trivial examples of non-contradictory ¬-paraconsistent logical systems. Furthermore, they are equipped with supplementing negations and bottom particles, and they are ¬-less-equivalent to classical propositional logic — so, that they will respect Principles (1), (2), (7), (8) and (9), but they will obviously not respect (3).

As it will be seen in the following, the hierarchy of logics C_n, $1 \leq n < \omega$ (cf. [da Costa, 1993] or [da Costa, 1974]) provide clear illustrations of **C**-systems based on classical logic. The cautious reader should bear in mind that C_ω (cf. Definition 32 below), the logic proposed as a kind of 'limit' for the hierarchy is *not* a **C**-system, not even an **LFI**. The real deductive limit for the hierarchy, the logic C_{Lim}, is an interesting example of a gently explosive **LFI** that is not finitely so, and it was studied in [Carnielli and Marcos, 1999]. The next definition will recall the hierarchy C_n, $1 \leq n < \omega$, in an axiomatic formulation of our own:

DEFINITION 23. Recall again the signature Σ from Remark 12. For every formula α, let $\circ\alpha$ be an abbreviation for the formula $\neg(\alpha \wedge \neg\alpha)$. The logic $C_1 = \langle For, \vdash_{C_1} \rangle$ can be axiomatized by the following schemas of a Hilbert calculus:

Axiom schemas:

(Ax1) $\alpha \to (\beta \to \alpha)$

(Ax2) $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$

(Ax3) $\alpha \to (\beta \to (\alpha \wedge \beta))$

(Ax4) $(\alpha \wedge \beta) \to \alpha$

(Ax5) $(\alpha \wedge \beta) \to \beta$

(Ax6) $\alpha \to (\alpha \vee \beta)$

(Ax7) $\beta \to (\alpha \vee \beta)$

(Ax8) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

(Ax9) $\alpha \vee (\alpha \to \beta)$

(Ax10) $\alpha \vee \neg\alpha$

(Ax11) $\neg\neg\alpha \to \alpha$

(bc1) $\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$

(ca1) $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \wedge \beta)$

(ca2) $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \vee \beta)$

(ca3) $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \to \beta)$

Inference rule:

(MP) $\dfrac{\alpha,\ \alpha \to \beta}{\beta}$

In general, given a set of axioms and rules of a logic **L**, we write $\Gamma \vdash_{\mathbf{L}} \alpha$ to say that there is proof in **L** of α from the premises in Γ. The subscript will be omitted when obvious from the context. If Γ is empty we say that α is a *theorem*.

The logic C_1 is a **C**-system based on classical logic such that $\bigcirc(p) = \{\circ p\} = \{\neg(p \wedge \neg p)\}$. We shall see that axioms (bc1), and (ca1)–(ca3) can be fully recycled by taking $\circ$ as a primitive connective instead of as an abbreviation.

Let α^1 abbreviate the formula $\neg(\alpha\wedge\neg\alpha)$, and α^{n+1} abbreviate the formula $(\neg(\alpha^n \wedge \neg\alpha^n))^1$. Then, each logic C_i of the hierarchy $C_n, 1 \leq n < \omega$ can be obtained by assuming $\bigcirc(p) = \{p^1, \ldots, p^i\}$. This is equivalent, of course, to setting $\circ\alpha \stackrel{\text{def}}{=} \alpha^1 \wedge \ldots \wedge \alpha^i$ in axioms (bc1) and (ca1)–(ca3). It is immediate that every logic C_i is a **C**-system based on classical logic. It is well-known that each C_i properly extends each C_{i+1}. ■

REMARK 24. Let the signature Σ^+ denote the signature Σ without the symbol $\neg$, and For^+ be the corresponding $\neg$-free fragment of For. *Positive classical logic*, from now on denoted by $\mathbf{CPL}^+$, can be axiomatized in the signature Σ^+ by the axioms (Ax1)–(Ax9) and (MP). *Classical propositional logic*, from now on denoted by **CPL**, is an extension of $\mathbf{CPL}^+$ in the signature Σ, where $\neg$ is governed by two dual axioms, (Ax10) and the following 'explosion law':

(exp) $\alpha \to (\neg\alpha \to \beta)$

That axiomatization should come as no surprise. Recall the notion of a classical negation from Definition 7. Clearly, for any logic **L** extending $\mathbf{CPL}^+$, the positive fragment of classical logic, a (primitive or defined) unary connective $\div$ of **L** is a classical negation iff the schemas $(\alpha \vee \div\alpha)$ and $(\alpha \to (\div\alpha \to \beta))$ are provable.

CPL is also the minimal consistent extension of C_1. Indeed, an alternative way of axiomatizing **CPL** is by adding $\circ\alpha$ to C_1 as a new axiom schema, and (exp) then follows from (bc1) and this new axiom, by (MP). *Positive intuitionistic logic* can be axiomatized from $\mathbf{CPL}^+$ by dropping (Ax9).

As is well-known (cf. [Mendelson, 1997]), any logic having (Ax1) and (Ax2) as axioms, and *modus ponens* (MP) as the only inference rule has a deductive implication.[6] As usual, bi-implication $\leftrightarrow$ will be defined by $(\alpha \leftrightarrow \beta) \stackrel{\text{def}}{=} ((\alpha \to \beta) \wedge (\beta \to \alpha))$. Note that, in the presence of a deductive implication $\to$, $\vdash (\alpha \leftrightarrow \beta)$ if, and only if, $\alpha \vdash \beta$ and $\beta \vdash \alpha$, that is, iff α and β are equivalent. Nevertheless, the equivalence of two formulas, in the logics we will study here, does not necessarily guarantee that these formulas can be freely intersubstituted everywhere, as we shall see below. ■

In any logic endowed with a deductive implication, the Principle of Explosion, (3), and the explosion law, (exp), are interderivable. So, for any such logic, if paraconsistency is to be obtained, (exp) must fail.

Recall that the definition of a **C**-system (Definition 22) mentioned **LFI**s in which the set $\bigcirc(p)$ could be taken as a singleton. The easiest way of realizing this intuition is by extending the original language of our logics so as to count from the start with a primitive connective $\circ$ for consistency (recall the signature Σ° from Remark 12). Sometimes this new connective can be dismissed, as in the case of the following class of logics:

DEFINITION 25. A **dC**-system is a **C**-system definable over the signature Σ. This means that in a **dC**-system $\circ$ can be explicitly defined in

[6] This is not always true, though, for logics extending (Ax1), (Ax2) and (MP) by the addition of new inference rules.

terms of the other connectives. ■

The next example and the subsequent theorem will show that **dC**-systems are even more ubiquitous than one might initially imagine.

EXAMPLE 26. Let $\Sigma^{\Diamond\Box}$ be the signature obtained by the addition of the new unary connectives $\Diamond$ and $\Box$ to the signature Σ, where the connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$ of Σ are interpreted as in classical logic and the new connectives are interpreted as in normal modal logics. So, $\Diamond\alpha$ (respectively, $\Box\alpha$) will be true in a given world iff α is true in some (respectively, any) world accessible to the former. The most obvious degenerate examples of normal modal logics are characterized by frames that are such that every world can access only itself or no other world. As shown in [Marcos, 2004a], inside any non-degenerate normal modal logic, a paraconsistent negation $\smile$ can be defined by setting $\smile\alpha \stackrel{\text{def}}{=} \Diamond\neg\alpha$, and a consistency connective can be defined by setting $\circ\alpha \stackrel{\text{def}}{=} \alpha \rightarrow \Box\alpha$.

Conversely, consider the signature Σ°, where the primitive negation $\neg$ is interpreted now over Kripke structures so as to behave exactly like the above connective $\smile$, that is, it is such that, for worlds x and y of a model $\mathcal{M}$ with an accessibility relation R:

$$\models_x^{\mathcal{M}} \neg\alpha \;\text{ iff }\; (\exists y)(x\mathsf{R}y \text{ and } \not\models_y^{\mathcal{M}} \alpha).$$

Moreover, let the consistency connective be interpreted in such a way that:

$$\models_x^{\mathcal{M}} \circ\alpha \;\text{ iff }\; \models_x^{\mathcal{M}} \alpha \text{ implies } (\forall y)(\text{if } x\mathsf{R}y \text{ then } \models_y^{\mathcal{M}} \alpha).$$

In the present case one can still redefine the previous connectives of $\Sigma^{\Diamond\Box}$. Indeed, one can define a bottom $\bot$ by setting $\bot \stackrel{\text{def}}{=} \alpha \wedge (\neg\alpha \wedge \circ\alpha)$, for an arbitrary formula α, and then define a classical negation $\sim$ by setting $\sim\alpha \stackrel{\text{def}}{=} \alpha \rightarrow \bot$. The original modal connectives can finally be defined by setting $\Diamond\alpha \stackrel{\text{def}}{=} \neg\sim\alpha$ and $\Box\alpha \stackrel{\text{def}}{=} \sim\neg\alpha$.

The above arguments show that any non-degenerate normal modal logic can be naturally recast in the signature of an **LFI**. In that sense, modal logics are typically paraconsistent. ■

THEOREM 27.
(i) **LFI1** (see Example 14) is a **C**-system, but not a **dC**-system.

(ii) $\mathbf{P}^1$ (see Example 15) is a **dC**-system.

(iii) The logics $C_n, 1 \leq n < \omega$, (see Definition 23) are all **dC**-systems.

(iv) Jaśkowski's Discussive Logic **D2** (see Example 19) is a **dC**-system.

(v) The normal modal logics from Example 26 are all **dC**-systems.

Proof. For item (i), observe first that **LFI1** is a **C**-system based on classical logic. Indeed, all axioms of $\mathbf{CPL}^+$ are validated by the 3-valued matrices of **LFI1**, and (MP) preserves validity. Second, the connective $\circ$ expresses consistency in **LFI1**, and **LFI1** is indeed a conservative extension of *Pac* obtained exactly by the addition of this connective. Finally, recall from Theorem 20 that *Pac* is not an **LFI**, and observe that $\circ$ is not definable from the other connectives of **LFI1**.

Items (ii)–(v) were already explained when the corresponding logics were introduced. ■

All **LFI**s studied from the next subsection on, unless explicit mention to the contrary, are **C**-systems based on classical logic, and thus can be axiomatized starting from $\mathbf{CPL}^+$.

*3.2 Towards **mbC**, a fundamental **LFI***

Before introducing our weakest **LFI** based on classical logic, we will introduce a very weak non-gently explosive paraconsistent logic.

DEFINITION 28. Recall from Remark 24 that $\neg$ in **CPL** was axiomatized by the addition of two dual clauses, (Ax10) and (exp). The paraconsistent logic *PI*, investigated in [Batens, 1980], extends $\mathbf{CPL}^+$ in the signature Σ (see Remark 24) by the addition of (Ax10). ■

It is worth noting that, because of (Ax10), one can count on *proof-by-cases*:

THEOREM 29. If $(\Gamma, \alpha \vdash_{PI} \beta)$ and $(\Delta, \neg\alpha \vdash_{PI} \beta)$ then $(\Gamma, \Delta \vdash_{PI} \beta)$. ■

Here are some other important properties of *PI*:

THEOREM 30. (i) *PI* is boldly paraconsistent.

For any boldly paraconsistent extension of *PI*:

(ii) *Reductio ad absurdum* is not a valid rule, i.e.:
$(\Delta, \beta \vdash \alpha)$ and $(\Pi, \beta \vdash \neg\alpha)$ implies $(\Delta, \Pi \vdash \neg\beta)$, and
$(\Delta, \neg\beta \vdash \alpha)$ and $(\Pi, \neg\beta \vdash \neg\alpha)$ implies $(\Delta, \Pi \vdash \beta)$
cannot obtain.

(iii) If $\rightarrow$ is a deductive implication, contraposition is not a valid rule, i.e.:
$\Gamma, \alpha \rightarrow \beta \vdash \neg\beta \rightarrow \neg\alpha$
$\Gamma, \alpha \rightarrow \neg\beta \vdash \beta \rightarrow \neg\alpha$
$\Gamma, \neg\alpha \rightarrow \beta \vdash \neg\beta \rightarrow \alpha$
$\Gamma, \neg\alpha \rightarrow \neg\beta \vdash \beta \rightarrow \alpha$
cannot obtain.

Proof. For item (i), note that *PI* has a deductive implication and is a fragment of both *Pac* and $\mathbf{P}^1$. Indeed, the axioms of *PI* are all validated by the matrices of *Pac* and by the matrices of $\mathbf{P}^1$, and (MP) preserves validity. Recall that those 3-valued extensions of *PI* were already proven to be boldly paraconsistent in Theorem 16.
For item (ii), let $\Delta = \Pi = \{\alpha, \neg\alpha\}$. Then, by *reductio*, the logic would be partially explosive.

For item (iii), using the properties of the deductive implication, we have that $\gamma \vdash \alpha \to \gamma$. Then again, by contraposition, the logic would turn out partially explosive. ■

As we will soon see (cf. Theorem 40), the upgrade of non-gently explosive logics into **LFI**s will help remedy the above mentioned deductive weaknesses, so typical of paraconsistent logics in general.

Here again, using the fact that *PI* is a deductive fragment of *Pac*, it can also be easily checked that:

THEOREM 31. The logic *PI*:
(i) does not have a supplementing negation, nor a bottom particle;
(ii) is not finitely trivializable;
(iii) is not an **LFI**. ■

Before proceeding, this seems a convenient place to mention some logics that live very close to *PI*:

DEFINITION 32. The logic C_{min} (cf. [Carnielli and Marcos, 1999]) is obtained from *PI* by the addition of $\neg\neg\alpha \to \alpha$ as a new axiom. The logic C_ω (cf. [da Costa, 1993]) is obtained from C_{min} by dropping (Ax9). Finally, the logic *CAR* (cf. [da Costa and Béziau, 1993]) is obtained from *PI* by adding $\alpha \to (\neg\alpha \to \neg\beta)$ as a new axiom. ■

Finally, here are some other important facts about *PI*:

THEOREM 33.
(i) *PI* does not prove any negated formula (that is, any formula of the form $\neg\delta$).
(ii) No two different negated formulas of *PI* are equivalent, i.e., $\neg\alpha \dashv\vdash_{PI} \neg\beta$ fails for any pair of distinct formulas α and β.

Proof. Item (i) was already proven in [Carnielli and Marcos, 1999] for C_{min}. Item (ii) was proven in [Urbas, 1989] for C_ω, and the proof can be easily adapted for *PI*. ■

As we saw in Theorem 31(iii), *PI* is not an **LFI**. We will now make the most obvious upgrade of *PI* that will turn it into an **LFI**, endowing it with the simplest axiomatic form of the principle (10), the Finite Gentle Principle of Explosion:

DEFINITION 34. Recall the signature Σ° from Remark 12 and the logic *PI* from Definition 28. The logic **mbC** is obtained from *PI*, over Σ°, by the addition of the following axiom schema:

(bc1) $\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$ ■

Notice that a particular form of axiom (bc1) had already been considered in Definition 23, but there $\circ\alpha$ was an abbreviation for $\neg(\alpha \wedge \neg\alpha)$, instead of a primitive connective. We recall that the intended reading of $\circ\alpha$ is 'α is consistent'. As we shall see, in general, $\circ\alpha$ is logically independent from $\neg(\alpha \wedge \neg\alpha)$.

If $\vdash_{\mathbf{mbC}}$ denotes the consequence relation of **mbC**, then we obtain, by (MP), the following:

$$\circ\alpha,\ \alpha,\ \neg\alpha \vdash_{\mathbf{mbC}} \beta \tag{11}$$

Rule (11) can be read as saying that 'if α is consistent and contradictory, then it explodes'. Clearly, this rule amounts to a realization of the Finite Gentle Principle of Explosion (10), as in our formulation of da Costa's C_n (Definition 23), with the difference that now $\circ$ is a primitive unary connective and *not* an abbreviation depending on conjunction and negation.

THEOREM 35. **mbC** is an **LFI**. In fact, it is a **C**-system based on **CPL**.

Proof. Note that **mbC** is indeed a fragment of **LFI1** and of $\mathbf{P}^1$, and in Theorem 20 the latter were shown to be **LFI**s. Moreover, we now know from rule (11) that the principle (9) holds in **mbC** (in fact its finite form (10) already holds). By design, we also know that **mbC** contains $\mathbf{CPL}^+$. Thus, **mbC** is a **C**-system based on **CPL** such that $\bigcirc(p) = \{\circ p\}$. ■

REMARK 36. It is easy to define supplementing negations in **mbC**. Consider first a negation $\wr$ set by $\wr\alpha \stackrel{\text{def}}{=} (\neg\alpha \wedge \circ\alpha)$. Notice that, as a particular instance of Theorem 10(i), $\bot \stackrel{\text{def}}{=} (\beta \wedge \wr\beta)$ defines a bottom particle, for an arbitrary β. Consider then a negation $\sim$ set by $\sim\alpha \stackrel{\text{def}}{=} \alpha \to \bot$. Clearly, $\forall\alpha\forall\beta(\alpha, \wr\alpha \vdash_{\mathbf{mbC}} \beta)$ and $\forall\alpha\forall\beta(\alpha, \sim\alpha \vdash_{\mathbf{mbC}} \beta)$.

In Remark 67, the semantic tools of Subsection 3.4, granting sound and complete possible-translations interpretations for **mbC**, will help showing that neither $\sim\alpha$ nor $\wr\alpha$ are always bottom particles. Moreover, these supplementing negations will in fact be seen to be inequivalent: though $\wr\alpha$ derives $\sim\alpha$, the converse is not true. While $\sim$ defines a classical negation, $\wr$ fails to be complementing. ■

REMARK 37. In spite of the term 'Logics of Formal *In*consistency', we have mentioned but a *consistency* connective $\circ$ this far. But **mbC** also has the dual *inconsistency* connective $\bullet$. To define it, one might make use of a classical negation, such as the negation $\sim$ defined in the above remark, and set $\bullet\alpha \stackrel{\text{def}}{=} \sim\circ\alpha$. ■

The logic **mbC** inherits the positive properties of *PI* (such as those of classical conjunction, or the deductive implication), but above we have seen that the former is much richer than the latter. As another illustration of this fact, from Theorem 35 and Remark 36 we can immediately see that none of the claims from Theorem 31 are true in **mbC**. Furthermore, the claims of Theorem 33 also do not hold for **mbC**:

THEOREM 38.
(i) There are in **mbC** theorems of the form $\neg\delta$, for some formula δ.

(ii) There are formulas α and β in **mbC** such that $\alpha \neq \beta$, α and β are equivalent, and $\neg\alpha$ and $\neg\beta$ are also equivalent.

Proof. (i) Consider any bottom particle $\bot$ of **mbC**. Then $(\bot \vdash_{\mathbf{mbC}} \neg\bot)$ and $(\neg\bot \vdash_{\mathbf{mbC}} \neg\bot)$, thus $\vdash_{\mathbf{mbC}} \neg\bot$, by Theorem 29.
(ii) Take α and β to be any two syntactically distinct bottom particles. ■

Even if, differently from *PI*, **mbC** does have negated theorems, it does not have consistent theorems:

THEOREM 39. There are in **mbC** no theorems of the form $\circ\delta$.

Proof. Use the classical matrices over $\{0, 1\}$ for $\wedge, \vee, \rightarrow$ and $\neg$, and pick for $\circ$ a matrix with value constant and equal to 0. ■

The price to pay for paraconsistency is that we necessarily lose some theorems and inferences dependent on the 'consistency presupposition'. This is illustrated, for instance, in Theorem 30, where *PI* and its extensions (satisfying certain assumptions) were shown to lack some usual classical rules such as *reductio* and contraposition. This loss in inferential power can be remedied in the **LFI**s exactly by adding convenient consistency assumptions at the object-language level. Indeed, some restricted forms of those rules can be proven in **mbC**:

THEOREM 40. The following *reductio* rules hold in **mbC**:

(i) $(\Gamma \vdash_{\mathbf{mbC}} \circ\alpha)$ and $(\Delta, \beta \vdash_{\mathbf{mbC}} \alpha)$ and $(\Lambda, \beta \vdash_{\mathbf{mbC}} \neg\alpha)$ implies $(\Gamma, \Delta, \Lambda \vdash_{\mathbf{mbC}} \neg\beta)$

(ii) $(\Gamma \vdash_{\mathbf{mbC}} \circ\alpha)$ and $(\Delta, \neg\beta \vdash_{\mathbf{mbC}} \alpha)$ and $(\Lambda, \neg\beta \vdash_{\mathbf{mbC}} \neg\alpha)$ implies $(\Gamma, \Delta, \Lambda \vdash_{\mathbf{mbC}} \beta)$

The following contraposition rules hold in **mbC**:

(iii) $\circ\beta, (\alpha \rightarrow \beta) \vdash_{\mathbf{mbC}} (\neg\beta \rightarrow \neg\alpha)$

(iv) $\circ\beta, (\alpha \rightarrow \neg\beta) \vdash_{\mathbf{mbC}} (\beta \rightarrow \neg\alpha)$

(v) $\circ\beta, (\neg\alpha \rightarrow \beta) \vdash_{\mathbf{mbC}} (\neg\beta \rightarrow \alpha)$

(vi) $\circ\beta, (\neg\alpha \rightarrow \neg\beta) \vdash_{\mathbf{mbC}} (\beta \rightarrow \alpha)$ ■

The last result is an instance of a more general phenomenon: Any classical rule can be recovered within our **C**-systems based on classical logic (check Subsection 3.6).

Intuitively, a contradiction might be seen as a sufficient condition for inconsistency. Indeed, here are some properties that relate the new connective of consistency to the more familiar connectives of $\mathbf{CPL}^+$:

THEOREM 41. In **mbC** the following hold:

(i) $\alpha, \neg\alpha \vdash_{\mathbf{mbC}} \neg\circ\alpha$

(ii) $(\alpha \wedge \neg\alpha) \vdash_{\mathbf{mbC}} \neg\circ\alpha$

(iii) $\circ\alpha \vdash_{\mathbf{mbC}} \neg(\alpha \wedge \neg\alpha)$

(iv) $\circ\alpha \vdash_{\mathbf{mbC}} \neg(\neg\alpha \wedge \alpha)$

The converses of these rules do not hold in **mbC**.

Proof. Items (i)–(iv) are easy consequences of the restricted forms of *reductio* from Theorem 40.

In order to prove the second part of the theorem, consider the matrices of $\mathbf{P}^1$ (Example 15), but substitute the matrix for negation by the 3-valued matrix for classical negation, $\sim$, from Example 14. ■

The last result hints to the fact that paraconsistent logics may easily have certain unexpected asymmetries. That's what happens, for instance, with da Costa's C_1. As we shall see, the converse of (iii) holds good in C_1, while the converse of (iv) fails, so that $\neg(\alpha\wedge\neg\alpha)$ and $\neg(\neg\alpha\wedge\alpha)$ are not equivalent formulas in C_1. Other even more shocking examples of asymmetries are the following:

THEOREM 42. In **mbC**:

(i) $(\alpha \wedge \beta) \dashv\vdash_{\mathbf{mbC}} (\beta \wedge \alpha)$ holds,
but $\neg(\alpha \wedge \beta) \dashv\vdash_{\mathbf{mbC}} \neg(\beta \wedge \alpha)$ does not.

(ii) $(\alpha \vee \beta) \dashv\vdash_{\mathbf{mbC}} (\beta \vee \alpha)$ holds,
but $\neg(\alpha \vee \beta) \dashv\vdash_{\mathbf{mbC}} \neg(\beta \vee \alpha)$ does not.

(iii) $(\alpha \wedge \neg\alpha) \dashv\vdash_{\mathbf{mbC}} (\neg\alpha \wedge \alpha)$ holds,
but $\neg(\alpha \wedge \neg\alpha) \dashv\vdash_{\mathbf{mbC}} \neg(\neg\alpha \wedge \alpha)$ does not.

(iv) $\gamma \vee \neg\gamma$ is a top particle, thus $(\alpha \vee \neg\alpha) \dashv\vdash_{\mathbf{mbC}} (\beta \vee \neg\beta)$ holds good.
But $\neg(\alpha \vee \neg\alpha) \dashv\vdash_{\mathbf{mbC}} \neg(\beta \vee \neg\beta)$ does not hold.

(v) $\alpha \dashv\vdash_{PI} (\neg\alpha \to \alpha)$ is a sound equivalence,
but $\neg\alpha \dashv\vdash_{\mathbf{mbC}} \neg(\neg\alpha \to \alpha)$ is not.

Proof. Using PI it is easy to prove the first parts of each item.

Items (i) to (iii). In order to check that none of the other parts hold, we can use again the matrices of $\mathbf{P}^1$ (Example 15), but redefining $(1 \wedge \frac{1}{2}) = (1 \vee \frac{1}{2}) = \frac{1}{2}$.

For item (iv), use the matrices of **LFI1** (Example 14), and take $v(p) \neq v(q)$ such that $v(p), v(q) \in \{1, \frac{1}{2}\}$. For item (v), use again the matrices of $\mathbf{P}^1$, and take $v(p) = \frac{1}{2}$. ■

REMARK 43. The last theorem illustrates the failure of the so-called *replacement property*, which states that, for every formula $\varphi(p_0, \ldots, p_n)$ and formulas $\alpha_0, \ldots, \alpha_n$, $\beta_0, \ldots, \beta_n$:

$$\text{(RP)}\quad (\alpha_0 \dashv\vdash \beta_0) \text{ and } \ldots \text{ and } (\alpha_n \dashv\vdash \beta_n) \text{ implies } \varphi(\alpha_0, \ldots, \alpha_n) \dashv\vdash \varphi(\beta_0, \ldots, \beta_n)$$

For example, using (RP), from $\alpha \dashv\vdash \beta$ we would immediately derive $\neg\alpha \dashv\vdash \neg\beta$. This does not hold for **mbC**. Recall, by the way, that $\alpha \dashv\vdash_{\mathbf{mbC}} \beta$

amounts to $\Vdash_{\mathbf{mbC}} \alpha \leftrightarrow \beta$, given the definition of bi-implication and the presence of a deductive implication in **mbC**.

Let $\varphi[p]$ denote a formula φ where the atomic formula p occurs, and $\varphi[p/\alpha]$ denote the formula obtained from φ by uniformly substituting all occurrences of p by the formula α. Given a pair of formulas α and β, we will say that they are *logically indistinguishable* if, for every formula $\varphi[p]$, we have that $\varphi[p/\alpha] \dashVdash \varphi[p/\beta]$. Clearly, a logic for which (RP) holds good is a logic for which every pair of equivalent formulas is logically indistinguishable. However, some formulas of a given logic may be logically indistinguishable, and yet this logic fail (RP).

Logics enjoying (RP) are called *self-extensional* in [Wójcicki, 1988]. Paradigmatic examples of such logics are given by normal modal logics. ■

We will show below that various other classes of **LFI**s do not enjoy the replacement property (see Theorems 44, 78 and 124).

A natural question here is whether our logics can be upgraded so as to recover the interesting property (RP) in the paraconsistent territory. To ensure that (RP) is obtainable in extensions of *PI* in the signature Σ, it is enough to add the rule:

(EC) $\forall\alpha\forall\beta((\alpha \dashVdash \beta)$ implies $(\neg\alpha \dashVdash \neg\beta))$

In [Urbas, 1989] paraconsistent extensions of C_ω (see Definition 32) enjoying the rule (EC) are shown to exist. The argument can be easily adapted to several extensions of *PI*, but fails for many other such extensions, as it will be shown below. In [da Costa and Béziau, 1993], the logic *CAR* (see Definition 32) was introduced as an extension of *PI* where (RP) holds good. But *CAR* is not an **LFI**, and it is not boldly paraconsistent, being partially explosive exactly as the Minimal Intuitionistic Logic *MIL* from Example 9. To obtain the replacement property in extensions of **mbC**, in the signature Σ°, a further rule is needed, namely:

(EO) $\forall\alpha\forall\beta((\alpha \dashVdash \beta)$ implies $(\circ\alpha \dashVdash \circ\beta))$

Before ending this subsection, let's quickly analyze the possible validity of (RP) in paraconsistent extensions of **mbC**, or in some of its fragments:

THEOREM 44. The replacement property (RP) cannot hold in any paraconsistent extension of **mbC** in which:
(i) $\circ{\div}{\div}\alpha$ holds, for some given classical negation $\div$; or
(ii) $\neg(\alpha \to \beta) \vdash (\alpha \wedge \neg\beta)$ holds.
The replacement property (RP) cannot hold in any left-adjunctive paraconsistent extension of *PI* in which:
(iii) $(\alpha \wedge \beta) \dashv\vdash \neg(\neg\alpha \vee \neg\beta)$ holds.
The replacement property (RP) cannot hold in any left-adjunctive paraconsistent logic in which:
(iv) $\neg(\alpha \wedge \neg\alpha)$ holds and $(\alpha \wedge \neg\alpha) \dashv\vdash \neg\neg(\alpha \wedge \neg\alpha)$.

Proof. (i) Since $\div$ is a classical negation, $\alpha \dashv\vdash \div\div\alpha$ and then, by (RP), we infer that $\circ\alpha \dashv\vdash \circ\div\div\alpha$. But $\circ\div\div\alpha$ is a theorem of the given logic, by hypothesis, then $\circ\alpha$ is a theorem. From (bc1), the logic turns out to be explosive with respect to the original negation $\neg$.
(ii) Consider the supplementing negation $\sim\alpha = (\alpha \to \bot)$ for **mbC**, where $\bot = (\alpha \wedge (\neg\alpha \wedge \circ\alpha))$, proposed in Remark 36. This negation was shown to be classical in Theorem 67. Then $\neg\sim\alpha \vdash (\alpha \wedge \neg\bot)$, by hypothesis, and so $\neg\sim\alpha \vdash \alpha$, using (Ax4). Since $\alpha, \sim\alpha \vdash \beta$ for every α and β then $\neg\sim\alpha, \sim\alpha \vdash \beta$ for every α and β, that is, the logic is controllably explosive in contact with $\sim p$. In particular, $\neg\sim\sim\alpha, \sim\sim\alpha \vdash \beta$ for every α and β. But $\alpha \dashv\vdash \sim\sim\alpha$ for a classical negation and so, using (RP), we may conclude that $\neg\alpha \dashv\vdash \neg\sim\sim\alpha$ and then $\neg\alpha, \alpha \vdash \beta$ for every β. That is, the logic will be explosive, not paraconsistent (with respect to the original negation $\neg$).
(iii) Since $(\neg\alpha \vee \neg\neg\alpha)$ is a theorem of *PI* then $\neg(\neg\alpha \vee \neg\neg\alpha) \dashv\vdash \neg(\neg\beta \vee \neg\neg\beta)$, for every α and β, by (RP). By hypothesis we infer that $(\alpha \wedge \neg\alpha) \dashv\vdash (\beta \wedge \neg\beta)$, so, by adjunction, we conclude in particular that $\alpha, \neg\alpha \vdash \beta$.
(iv) Since $\neg(\alpha \wedge \neg\alpha)$ is a theorem, by hypothesis, then $\neg\neg(\alpha \wedge \neg\alpha) \dashv\vdash \neg\neg(\beta \wedge \neg\beta)$ for every α and β, by (RP). Then $(\alpha \wedge \neg\alpha) \dashv\vdash (\beta \wedge \neg\beta)$, by hypothesis. The result follows now as in item (iii). ■

REMARK 45. To obtain paraconsistent extensions of **mbC** validating both (EC) and (EO) is a feasible task. Examples of such logics were already offered in Example 26: Notice indeed that axiom (bc1) and rules (EC) and (EO) are all satisfied by the minimal normal modal logic K, thus also by any of its normal modal extensions. ■

*3.3 Valuation semantics for **mbC***

In this subsection we provide an interpretation for **mbC** in terms of valuation semantics. This example will help to clarify the connections with other semantical interpretations, as well as to devise relevant open problems towards obtaining a theoretical framework for further investigation in the foundations of paraconsistent logic.

At the beginning of their history, most **C**-systems were introduced exclusively in proof-theoretical terms (see, for a survey, [Carnielli and Marcos, 2002]). Only some years later adequate semi-truth-functional bivalued semantics were proposed to interpret them. We have mentioned above several 3-valued samples of paraconsistent logics (Examples 13, 14 and 15), and many examples of paraconsistent logics with a modal semantics (Example 26). We have also shown above that the logic **mbC**, our weakest **LFI** based on classical logic, fails the replacement property. The next results, adapted from [Marcos, 2004d] (the first theorem is inspired by [Carnielli, 1990], the second by [Arruda, 1975]) and abstracting from well-known arguments by K. Gödel and J. Dugundji about uncharacterizability of, respectively, intuitionistic and normal modal logics by finite models, obtain sufficient conditions for showing that **mbC** fails to be characterizable by finite-valued truth-functional matrices.

THEOREM 46. Consider the following infinite-valued matrix over Σ° whose truth-values are the ordinals in $\omega + 1 = \omega \cup \{\omega\}$, of which all elements in ω are designated. The connectives of Σ° are interpreted using maps $v : For^\circ \longrightarrow \omega + 1$ such that:

$$\begin{array}{lll} v(\alpha \wedge \beta) & = & 0, \text{ if } v(\beta) = v(\alpha)+1 \text{ or } v(\alpha) = v(\beta)+1; \\ & & max(v(\alpha), v(\beta)), \text{ otherwise}; \\ v(\alpha \vee \beta) & = & min(v(\alpha), v(\beta)); \\ v(\alpha \to \beta) & = & \omega, \text{ if } v(\alpha) \in \omega \text{ and } v(\beta) = \omega; \\ & & v(\beta), \text{ if } v(\alpha) = \omega \text{ and } v(\beta) \in \omega; \\ & & 0, \text{ if } v(\alpha) = \omega \text{ and } v(\beta) = \omega; \\ & & max(v(\alpha), v(\beta)), \text{ otherwise}; \\ v(\neg\alpha) & = & \omega, \text{ if } v(\alpha) = 0; \\ & & 0, \text{ if } v(\alpha) = \omega; \\ & & v(\alpha)+1, \text{ otherwise}; \\ v(\circ\alpha) & = & \omega, \text{ if } 0 < v(\alpha) < \omega; \\ & & 0, \text{ otherwise}. \end{array}$$

Let **L** be any logic with a deductive implication defined over Σ° and extending $\mathbf{CPL}^+$ which is sound with respect to the infinite-valued matrices defined above. Then **L** cannot be semantically characterized by finite-valued matrices.

Proof. Let **L** be any logic satisfying the given hypothesis. Define the following formulas over Σ°:

$$\varphi_{ij} \stackrel{\text{def}}{=} \circ p_i \wedge p_i \wedge \neg p_j, \text{ for } 0 \leq i < j;$$

$$\psi_n \stackrel{\text{def}}{=} \bigvee_{0 \leq i < j \leq n} (\varphi_{ij} \to p_{n+1}), \text{ for } n > 0.$$

It is easy to see that all formulas ψ_n can be assigned the non-designated truth-value ω in the infinite model according to the given matrices: Just assign $v(p_i) = i$ for $0 \leq i \leq n$ and $v(p_{n+1}) = \omega$. On the other hand, the formula ψ_n must be a tautology in every m-valued set of matrices that is sound for **L** and such that $m < n$. Indeed, if $m < n$, by the Pigeonhole Principle of elementary combinatorics there exist i and j such that $v(p_i) = v(p_j)$. But the formula $(\circ\alpha \wedge \alpha \wedge \neg\alpha) \to \beta$ is a theorem of **L** for every α and β, then it must be evaluated as designated in every set of adequate matrices for **L**. The same occurs, of course, with the formula $\alpha \to (\alpha \vee \beta)$. It follows from the validity of such formulas (and by *modus ponens*) that ψ_n is also validated in every adequate set of m-valued matrices which is sound for **L** and such that $m < n$, although, as mentioned before, no formula ψ_n can be a theorem of **L**. Therefore, **L** cannot be semantically characterized by any collection of finite-valued matrices. ■

THEOREM 47. Consider next the following infinite-valued matrix over Σ° whose truth-values are the ordinals in ω, of which 0 is the only non-designated value. The connectives of Σ° are now interpreted using maps $v : For^\circ \longrightarrow \omega$ such that:

$$\begin{array}{lcl} v(\alpha \wedge \beta) & = & 1, \text{ if } v(\alpha) > 0 \text{ and } v(\beta) > 0; \\ & & 0, \text{ otherwise;} \\ v(\alpha \vee \beta) & = & 1, \text{ if } v(\alpha) > 0 \text{ or } v(\beta) > 0; \\ & & 0, \text{ otherwise;} \\ v(\alpha \to \beta) & = & 0, \text{ if } v(\alpha) > 0 \text{ and } v(\beta) = 0; \\ & & 1, \text{ otherwise;} \\ v(\neg\alpha) & = & 1, \text{ if } v(\alpha) = 0; \\ & & v(\alpha) - 1, \text{ otherwise;} \\ v(\circ\alpha) & = & 0, \text{ if } v(\alpha) > 1; \\ & & 1, \text{ otherwise;} \end{array}$$

Let **L** be any logic with a deductive implication defined over Σ° and extending $\mathbf{CPL}^+$ which is sound with respect to the infinite-valued matrices defined above. Then **L** cannot be semantically characterized by finite-valued matrices.

Proof. Let **L** be any logic satisfying the given hypothesis, and let $\neg^i$, for $i \geq 0$, denote i iterations of the negation $\neg$. Define the following formulas over Σ°:

$$\varphi_{ij} \stackrel{\text{def}}{=} \neg^i p \leftrightarrow \neg^j p, \text{ for } 0 \leq i < j.$$

It is easy to see that the above matrices invalidate all formulas φ_{ij}, assigning to them the non-designated truth-value 0: Just assign $v(p) = j$, and notice that $v(\neg^j p) = 0$ while $v(\neg^i p) > 0$. On the other hand, by the Pigeonhole Principle, any m-valued set of matrices that is sound and complete for **L** will be such that, given some i, there is some $i < j \leq (i + n^n)$, such that $v(\neg^i p) = v(\neg^j p)$, for all v. In that case, φ_{ij} is validated, an absurd. Therefore, **L** cannot be semantically characterized by any collection of finite-valued matrices. ■

As we shall see, the above results will help to show that many paraconsistent logics are not finite-valued. Either of them, in fact, allows us to prove, at this point:

COROLLARY 48. The logic **mbC** is not characterizable by finite matrices.

Proof. The logic **mbC** is sound with respect to both the matrices of Theorem 46 and those of Theorem 47. ■

Now we will show how a simple (semi-truth-functional) adequate valuation semantics can be attached to the logic **mbC**. In the next subsection, we will endow it with the much richer semantics of possible-translations. This new semantics, as we shall see, not only gives an interpretation to contradictory situations, but also offers an explanation for conflicting scenarios.

DEFINITION 49. Let $\mathbf{2} \stackrel{\text{def}}{=} \{0, 1\}$ be the set of two truth-values, where 1 denotes the 'true' value and 0 denotes the 'false' value. An **mbC**-valuation is any function $v : For^{\circ} \longrightarrow \mathbf{2}$ subject to the following clauses:

(v1) $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$;
(v2) $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$;
(v3) $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$;
(v4) $v(\neg\alpha) = 0$ implies $v(\alpha) = 1$;
(v5) $v(\circ\alpha) = 1$ implies $v(\alpha) = 0$ or $v(\neg\alpha) = 0$. ■

For a collection $\Gamma \cup \{\alpha\}$ of formulas of **mbC**, $\Gamma \vDash_{\mathbf{mbC}} \alpha$ means, as usual, that α is assigned the value 1 for every **mbC**-valuation which assigns value 1 to the elements of Γ. The proof of soundness for **mbC** with respect to **mbC**-valuations is immediate.

REMARK 50. Given clause (v5) in the above definition of a valuation semantics for **mbC**, it is clear that this logic does not admit a trivial model, that is, there is no v such that $v(\alpha) = 1$ for every formula α. In particular, given a trivial theory Γ of **mbC**, for every **mbC**-valuation v, then there is some $\gamma \in \Gamma$ such that $v(\gamma) = 0$, thus $v(\neg\gamma) = 1$, by clause (v4). This observation reveals a typical semantical feature of **LFI**s. Indeed, other non-gently explosive paraconsistent logics might well allow for such trivial models. For instance, the logic *Pac* (Example 13), despite being maximal relative to classical logic (cf. [Batens, 1980]), does admit for such a model: Just take $v(\alpha) = \frac{1}{2}$, where $\frac{1}{2}$ is a designated value. ■

THEOREM 51. [Soundness] Let $\Gamma \cup \{\alpha\}$ be a set of formulas in For°. Then: $\Gamma \vdash_{\mathbf{mbC}} \alpha$ implies $\Gamma \vDash_{\mathbf{mbC}} \alpha$.

Proof. Just check that all axioms of **mbC** assume only the value 1 in any **mbC**-valuation, and that (MP) preserves validity. ■

In order to prove completeness it is convenient to prove first some auxiliary lemmas. Let $\Delta \cup \{\alpha\}$ be a set of formulas in For°. We say that a theory Δ is *relatively maximal with respect to α in* **mbC** if $\Delta \nvdash_{\mathbf{mbC}} \alpha$ and for any formula β in For° such that $\beta \notin \Delta$ we have $\Delta, \beta \vdash_{\mathbf{mbC}} \alpha$. The usual Lindenbaum-Asser argument (cf. [Béziau, 1999]) shows that inside any compact Tarskian logic — such as **mbC** — every non-trivial theory can be extended into a relatively maximal theory:

LEMMA 52. Let **L** be a compact Tarskian logic over a signature $\widehat{\Sigma}$. Given some set of formulas Γ and a formula α such that $\Gamma \nVdash_{\mathbf{L}} \alpha$, then there is a set $\Delta \supseteq \Gamma$ that is relatively maximal with respect to α in **L**.

Proof. Consider an enumeration $\{\varphi_n\}_{n\in\mathbb{N}}$ of the formulas in $For_{\mathbf{L}}$, and a chain Γ_n, $n \in \mathbb{N}$, of sets built as follows:

$$\begin{array}{lll} \Delta_0 & = & \Gamma; \\ \Delta_{n+1} & = & \Delta_n \cup \{\varphi_n\}, \text{ if } \ \Delta_n, \varphi_n \nVdash_{\mathbf{L}} \alpha; \\ & & \Delta_n, \text{ otherwise.} \end{array}$$

Let $\Delta = \bigcup_{n\in\mathbb{N}} \Delta_n$. We will show that Δ is relatively maximal with respect to α in **L**. First of all, notice that, by an easy induction over the above chain, we can conclude that $\Delta_n \nVdash_{\mathbf{L}} \alpha$, for every $n \in \mathbb{N}$. Moreover, $\Delta \nVdash_{\mathbf{L}} \alpha$. Indeed, if that was not the case, by compactness there would be some finite $\Delta^{\mathsf{fin}} \subseteq \Delta$ such that $\Delta^{\mathsf{fin}} \Vdash_{\mathbf{L}} \alpha$. But then, using cut, there would be some $\Delta_m \supseteq \Delta^{\mathsf{fin}}$ such that $\Delta_m \Vdash_{\mathbf{L}} \alpha$, impossible. Now, consider some $\beta \notin \Delta$. Any such β is such that $\beta = \varphi_n$, for some n. Thus $\beta \notin \Delta_{n+1}$, given reflexivity and $\Delta_{n+1} \subseteq \Delta$. So, $\Delta_{n+1} = \Delta_n$ and $\Delta_n, \beta \Vdash_{\mathbf{L}} \alpha$, by construction. Once $\Delta_n \subseteq \Delta$, we must conclude by monotonicity that $\Delta, \beta \Vdash_{\mathbf{L}} \alpha$. ■

Now we can prove:

LEMMA 53. Any relatively maximal set of formulas is a closed theory.

Proof. We have to check that, given a set of formulas Δ that is relatively maximal with respect to a formula α, then $\Delta \vdash_{\mathbf{mbC}} \beta$ iff $\beta \in \Delta$. Right to left is obvious by (Con1). From left to right, given some $\beta \notin \Delta$ we have that (a) $\Delta \nvdash_{\mathbf{mbC}} \alpha$ and (b) $\Delta, \beta \vdash_{\mathbf{mbC}} \alpha$, since Δ is relatively maximal with respect to α. But then, from (a) and (b) we conclude that $\Delta \nvdash_{\mathbf{mbC}} \beta$, by (Con3). ■

LEMMA 54. Let $\Delta \cup \{\alpha\}$ be a set of formulas in For° such that Δ is relatively maximal with respect to α in **mbC**. Then:

(i) $(\beta \wedge \gamma) \in \Delta$ iff $\beta \in \Delta$ and $\gamma \in \Delta$.

(ii) $(\beta \vee \gamma) \in \Delta$ iff $\beta \in \Delta$ or $\gamma \in \Delta$.

(iii) $(\beta \to \gamma) \in \Delta$ iff $\beta \notin \Delta$ or $\gamma \in \Delta$.

(iv) $\beta \notin \Delta$ implies $\neg\beta \in \Delta$.

(v) $\circ\beta \in \Delta$ implies $\beta \notin \Delta$ or $\neg\beta \notin \Delta$.

Proof. Lemma 53 will be used to prove each of the above items. To simplify reference, call this result (RiC).
Item (i) is proved from (RiC), axioms (Ax3), (Ax4), (Ax5) and (MP).
(ii) from (RiC), axioms (Ax6), (Ax7), (Ax8) and (MP).
(iii) from (RiC), (ii), axioms (Ax1), (Ax9) and (MP).
(iv) from (RiC), axiom (Ax10) and (MP).
For item (v), suppose $\beta \in \Delta$ and $\neg\beta \in \Delta$. Then, from (RiC), (bc1) and relative maximality, we conclude that $\circ\beta \notin \Delta$. ■

COROLLARY 55. The characteristic function of a relatively maximal set of formulas in **mbC** defines an **mbC**-valuation.

Proof. Let Δ be a set of formulas relatively maximal with respect to α and define a function $v : For^\circ \longrightarrow \mathbf{2}$ such that, for any formula β in For°, $v(\beta) = 1$ iff $\beta \in \Delta$. Using the previous lemma it is easy to see that v satisfy clauses (v1) to (v5) of Definition 49. ■

THEOREM 56. [Completeness] Let $\Gamma \cup \{\alpha\}$ be a set of formulas in For°. Then: $\Gamma \vDash_{\mathbf{mbC}} \alpha$ implies $\Gamma \vdash_{\mathbf{mbC}} \alpha$.

Proof. Given a formula α in For° such that $\Gamma \nvdash_{\mathbf{mbC}} \alpha$ one may, by the Lindenbaum-Asser argument, extend Γ to a set Δ relatively maximal with respect to α. As $\Delta \nvdash_{\mathbf{mbC}} \alpha$, then $\alpha \notin \Delta$, because of (Con1). By Corollary 55, the characteristic function v of Δ is an **mbC**-valuation such that, for any $\beta \in \Delta$, $v(\beta) = 1$, while $v(\alpha) = 0$. So, $\Delta \nvDash_{\mathbf{mbC}} \alpha$, and in particular $\Gamma \nvDash_{\mathbf{mbC}} \alpha$. ■

An immediate application of the valuation semantics for **mbC** is the obtainment of easy semantical proofs of the following remarkable characteristics of **mbC**. The connectives $\wedge, \vee$ and $\rightarrow$ are not interdefinable as in the classical case.

THEOREM 57. The following rule holds in **mbC**:
(i) $(\neg\alpha \rightarrow \beta) \vdash_{\mathbf{mbC}} (\alpha \vee \beta)$,
but none of the following rules hold in **mbC**:
(ii) $(\alpha \vee \beta) \vdash_{\mathbf{mbC}} (\neg\alpha \rightarrow \beta)$;
(iii) $\neg(\neg\alpha \rightarrow \beta) \vdash_{\mathbf{mbC}} \neg(\alpha \vee \beta)$;
(iv) $\neg(\alpha \vee \beta) \vdash_{\mathbf{mbC}} \neg(\neg\alpha \rightarrow \beta)$;
(v) $(\alpha \rightarrow \beta) \vdash_{\mathbf{mbC}} \neg(\alpha \wedge \neg\beta)$;
(vi) $\neg(\alpha \wedge \neg\beta) \vdash_{\mathbf{mbC}} (\alpha \rightarrow \beta)$;
(vii) $\neg(\alpha \rightarrow \beta) \vdash_{\mathbf{mbC}} (\alpha \wedge \neg\beta)$;
(viii) $(\alpha \wedge \neg\beta) \vdash_{\mathbf{mbC}} \neg(\alpha \rightarrow \beta)$;
(ix) $\neg(\neg\alpha \wedge \neg\beta) \vdash_{\mathbf{mbC}} (\alpha \vee \beta)$;
(x) $(\alpha \vee \beta) \vdash_{\mathbf{mbC}} \neg(\neg\alpha \wedge \neg\beta)$;
(xi) $\neg(\neg\alpha \vee \neg\beta) \vdash_{\mathbf{mbC}} (\alpha \wedge \beta)$;
(xii) $(\alpha \wedge \beta) \vdash_{\mathbf{mbC}} \neg(\neg\alpha \vee \neg\beta)$. ■

EXAMPLE 58. The first **LFI** ever to receive an interpretation in terms of valuation semantics was the logic C_1 (cf. [da Costa and Alves, 1977]). The original set of clauses characterizing the C_1-valuations is the following:

(vC1) $v(\alpha_1 \wedge \alpha_2) = 1$ iff $v(\alpha_1) = 1$ and $v(\alpha_2) = 1$;

(vC2) $v(\alpha_1 \vee \alpha_2) = 1$ iff $v(\alpha_1) = 1$ or $v(\alpha_2) = 1$;

(vC3) $v(\alpha_1 \rightarrow \alpha_2) = 1$ iff $v(\alpha_1) = 0$ or $v(\alpha_2) = 1$;

(vC4) $v(\neg\alpha) = 0$ implies $v(\alpha) = 1$;

(vC5) $v(\neg\neg\alpha) = 1$ implies $v(\alpha) = 1$;

(vC6) $v(\circ\beta) = v(\alpha \to \beta) = v(\alpha \to \neg\beta) = 1$ implies $v(\alpha) = 0$;

(vC7) $v(\circ(\alpha\#\beta)) = 0$ implies $v(\circ\alpha) = 0$ or $v(\circ\beta) = 0$, where $\# \in \{\wedge, \vee, \to\}$;

where, as usual, $\circ\alpha$ denotes the formula $\neg(\alpha \wedge \neg\alpha)$. ■

We finish this subsection by stating some simple results in model theory for **LFI**s. From now on, until to the end of this subsection, **L** will denote a propositional logic presented through valuation semantics satisfying at least the clauses defining the **mbC**-valuations (Definition 49) and such that $\vDash_{\mathbf{L}}$ is compact.

THEOREM 59. [Interpolation Theorem for **LFI**s] Let φ and ψ be formulas in **L** such that $\varphi \vDash_{\mathbf{L}} \psi$. Then:

1. φ is a bottom particle, that is, equivalent to a formula of the form $\delta \wedge (\neg\delta \wedge \circ\delta)$; or
2. ψ is a top particle, that is, a tautology; or
3. there exists a formula γ (called *interpolant*), where every propositional variable occurring in γ occurs simultaneously in φ and in ψ, such that $\varphi \vDash_{\mathbf{L}} \gamma$ and $\gamma \vDash_{\mathbf{L}} \psi$.

Proof. The argument is similar to the classical one, just changing the classical bottom by $p\wedge(\neg p\wedge\circ p)$, where p is a propositional variable occurring simultaneously in φ and in ψ. ■

DEFINITION 60. Let P be a non-empty set of propositional variables. A theory Δ of **L** is **L**-*P-complete* if, for every formula δ in the language of **L** which uses exclusively variables in P, we have that $\Delta \vDash_{\mathbf{L}} \delta\wedge\circ\delta$ or $\Delta \vDash_{\mathbf{L}} \delta\wedge\neg\delta$ or $\Delta \vDash_{\mathbf{L}} \circ\delta \wedge \neg\delta$.

THEOREM 61. [Joint Consistency for **LFI**s] Let Δ_1 and Δ_2 be non-trivial theories of **L** and let P_i be the set of propositional variables occurring in Δ_i ($i \in \{1, 2\}$). Assume that Δ_i extend an **L**-$(P_1 \cap P_2)$-complete theory Δ ($i \in \{1, 2\}$). Suppose also that, for every formula α using propositional variables in P_2, $\Delta_2 \vDash_{\mathbf{L}} \alpha$ implies $\Delta_2 \nvDash_{\mathbf{L}} \neg\alpha$. Then Δ_1 and Δ_2 are jointly non-trivial, that is, $\Delta_1 \cup \Delta_2$ is non-trivial.

Proof. Assume the hypotheses of the theorem, and suppose further that $\Delta_1 \cup \Delta_2$ is trivial, that is, $\Delta_1, \Delta_2 \vDash_{\mathbf{L}} \beta$ for every formula β using propositional variables in $P_1 \cup P_2$. This means that $\Delta_1 \cup \Delta_2$ has no non-trivial models, therefore there are finite, non-empty sets $\Gamma_i \subseteq \Delta_i$ (for $i = 1, 2$) such that $\Gamma_1 \cup \Gamma_2$ has no non-trivial models, by compactness. Note that Γ_1 and Γ_2 are non-empty because Δ_1 and Δ_2 are non-trivial. Let α_1 and α_2 be the conjunction of the formulas in Γ_1 and Γ_2, respectively. Then $\alpha_1 \vDash_{\mathbf{L}} \neg\alpha_2$, in the light of Remark 50. By the Interpolation Theorem for **LFI**s, there is a formula α using exclusively propositional variables in $P_1 \cap P_2$ such that $\alpha_1 \vDash_{\mathbf{L}} \alpha$ and $\alpha \vDash_{\mathbf{L}} \neg\alpha_2$. Since $\Delta_1 \vDash_{\mathbf{L}} \alpha_1$ then $\Delta_1 \vDash_{\mathbf{L}} \alpha$. Suppose that $\Delta_1 \vDash_{\mathbf{L}} \neg\alpha$ and $\Delta_1 \vDash_{\mathbf{L}} \circ\alpha$. Then $\Delta_1 \vDash_{\mathbf{L}} \alpha \wedge (\neg\alpha \wedge \circ\alpha)$, an absurd (because Δ_1 has a non-trivial model and $\alpha \wedge (\neg\alpha \wedge \circ\alpha)$ is unsatisfiable). Therefore $\Delta_1 \nvDash_{\mathbf{L}} \neg\alpha$ or $\Delta_1 \nvDash_{\mathbf{L}} \circ\alpha$, and so $\Delta \nvDash_{\mathbf{L}} \neg\alpha$ or $\Delta \nvDash_{\mathbf{L}} \circ\alpha$, because Δ_1 extends Δ. Now, suppose that $\Delta \vDash_{\mathbf{L}} \alpha$. Since Δ_2 extends Δ we infer that $\Delta_2 \vDash_{\mathbf{L}} \alpha$. But $\alpha \vDash_{\mathbf{L}} \neg\alpha_2$ and so $\Delta_2 \vDash_{\mathbf{L}} \neg\alpha_2$. On the other hand $\Delta_2 \vDash_{\mathbf{L}} \alpha_2$, contradicting the hypothesis about Δ_2. Then $\Delta \nvDash_{\mathbf{L}} \alpha$. But in that case Δ is an $\mathbf{L}$-$(P_1 \cap P_2)$-complete theory and α is a formula with propositional variables in $P_1 \cap P_2$ such that $\Delta \vDash_{\mathbf{L}} \alpha$ fails, and one among $\Delta \vDash_{\mathbf{L}} \neg\alpha$ and $\Delta \vDash_{\mathbf{L}} \circ\alpha$ also fails, an absurd. Therefore $\Delta_1 \cup \Delta_2$ is $\mathbf{L}$-$(P_1 \cup P_2)$-complete. ■

*3.4 Possible-translations semantics for **LFI**s*

Notwithstanding the fact that the proof of completeness by means of valuations for **LFI**s is simple to obtain, this semantics does not do a good job in explaining intrinsic singularities of such logics. In particular, it is not obvious from the very definition of the valuation semantics for **mbC** (Definition 49) that this logic is decidable. A decision procedure can be obtained with some further effort, however, by adapting the well-known procedure of matrices into a procedure of 'quasi matrices' (check for instance [da Costa and Alves, 1977] and [da Costa *et al.*, 1995]). At any rate, valuation semantics may be very useful as a technical device that helps in simplifying the completeness proof with respect to possible-translations semantics that we present in this subsection, as well as in defining binary tableaux for our logics, as it will be illustrated in the next section. Possible-translations semantics were introduced in [Carnielli, 1990]; for a study of their scope and for formal definitions related to them check [Marcos, 2004c].

The notion of *translation* between a logic $\mathbf{L}1$ and a logic $\mathbf{L}2$ is essential here: It is simply a mapping $* : \mathbf{L}1 \longrightarrow \mathbf{L}2$ such that, for every set $\Gamma \cup \{\alpha\}$ of $\mathbf{L}1$-formulas,

$$\Gamma \vdash_{\mathbf{L1}} \alpha \ \text{ implies } \ \Gamma^* \vdash_{\mathbf{L2}} \alpha^*.$$

Here, α^* denotes $*(\alpha)$ and Γ^* stands for $\{\gamma^* \ : \ \gamma \in \Gamma\}$. If 'implies' is replaced by 'iff' in the definition above, then $*$ is called a *conservative translation*. See [da Silva *et al.*, 1999] and [Coniglio and Carnielli, 2002] for a general account of translations and conservative translations.

Consider now the following three-valued matrices, where T and t are the designated values:

$\wedge$	T	t	F
T	t	t	F
t	t	t	F
F	F	F	F

$\vee$	T	t	F
T	t	t	t
t	t	t	t
F	t	t	F

$\to$	T	t	F
T	t	t	F
t	t	t	F
F	t	t	t

	$\neg_1$	$\neg_2$	$\circ_1$	$\circ_2$
T	F	F	t	F
t	F	t	F	F
F	T	t	t	F

In order to provide interpretations to the connectives of **mbC** by means of possible-translations semantics we should first understand these tables. The truth-value t may be interpreted as 'true by default', or 'true by lack of evidence to the contrary', and T and F are, as usual, 'true' and 'false'. The matrices for conjunction, disjunction and implication never return the value T, so in principle one is never absolutely sure about the truth status of some compound sentences. There are two distinct interpretations for negation $\neg$ and for the consistency operator $\circ$. The basic intuition is the idea of *multiple scenarios* concerning the dynamics of evaluation of propositions: We may think that there are two kinds of situations concerning non-true propositions with respect to successive moments of time. In the first situation, a true-by-default proposition is treated as a true proposition with respect to the negation $\neg_1$. On the other situation, we can consider the case in which the negation of any other value than 'true' became true-by-default — this is expressed by the negation $\neg_2$. On what concerns the consistency operator $\circ$, the first interpretation $\circ_1$ only considers as true-by-default the 'classical' values T and F, while $\circ_2$ assigns falsehood to every truth-value.

This collection of truth-tables, which we call $\mathcal{M}_0$, will be used to give the desired semantics for **mbC**. Now, considering the algebra $For^0_{\mathcal{M}}$ of formulas generated by $\mathcal{P}$ over the signature of $\mathcal{M}_0$, let's define the set $\textsc{Tr}_0$ of all functions $* : For^\circ \longrightarrow For^0_{\mathcal{M}}$ subjected to the following clauses:

$(tr0)$ $p^* = p$, if $p \in \mathcal{P}$;

$(tr1)$ $(\alpha\#\beta)^* = (\alpha^*\#\beta^*)$, for all $\# \in \{\wedge, \vee, \to\}$;

$(tr2)$ $(\neg\alpha)^* \in \{\neg_1\alpha^*, \neg_2\alpha^*\}$;

$(tr3)$ $(\circ\alpha)^* \in \{\circ_1\alpha^*, \circ_2\alpha^*, \circ_1(\neg\alpha)^*\}$.

We say the pair $\text{PT}_0 = \langle \mathcal{M}_0, \textsc{Tr}_0 \rangle$ is a *possible-translations semantical structure for* **mbC**. If $\vDash^0_{\mathcal{M}}$ denotes the consequence relation in $\mathcal{M}_0$, and $\Gamma \cup \{\alpha\}$ is a set of formulas of **mbC**, the associated PT-consequence relation, $\models_{\text{PT}_0}$, is defined as:

$$\Gamma \models_{\text{PT}_0} \alpha \text{ iff } \Gamma^* \vDash^0_{\mathcal{M}} \alpha^* \text{ for all translations } * \text{ in } \textsc{Tr}_0.$$

We will call a *possible translation* of a formula α any image of it through some function in $\textsc{Tr}_0$. We can immediately prove the following:

THEOREM 62. [Soundness] Let $\Gamma\cup\{\alpha\}$ be a set of formulas of **mbC**. Then $\Gamma\vdash_{\mathbf{mbC}}\alpha$ implies $\Gamma\models_{\mathrm{PT}_0}\alpha$.

Proof. It is sufficient to check that the (finite) collection of all possible translations of each axiom produces tautologies in the matrices of $\mathcal{M}_0$ and that all possible translations of the rule (MP) preserve validity. The verification is immediate, and we leave it as exercise to the reader. ■

In order to prove completeness, our strategy is to show that each **mbC**-valuation v determines a translation $*$ and a three-valued valuation w defined in the usual way over the matrices $\mathcal{M}_0$ such that, for every formula α of **mbC**,

$$w(\alpha^*)\in\{T,t\}\ \text{ iff }\ v(\alpha)=1$$

and thus rely on the proof of valuation completeness for **mbC**.

DEFINITION 63. Let the mapping $l:\ For^{\circ}\longrightarrow\mathbb{N}$ be the *complexity measure* defined over the signature Σ°, by: $l(p)=0$, for $p\in\mathcal{P}$; $l(\varphi\#\psi)=l(\varphi)+l(\psi)+1$, for $\#\in\{\wedge,\vee,\rightarrow\}$; $l(\neg\varphi)=l(\varphi)+1$; and $l(\circ\varphi)=l(\varphi)+2$.■

The following result was taken from [Marcos, 2004d]:

THEOREM 64. [Representability] Given an **mbC**-valuation v there is a translation $*$ in TR_0 and a valuation w in $\mathcal{M}_0$ such that, for every formula α in **mbC**:

$$w(\alpha^*)=T\ \text{ implies }\ v(\neg\alpha)=0;\ \text{ and}$$
$$w(\alpha^*)=F\ \text{ iff }\ v(\alpha)=0.$$

Proof. For $p\in\mathcal{P}$ define the valuation w as follows:

$$\begin{array}{lll} w(p)=F & \text{if} & v(p)=0;\\ w(p)=T & \text{if} & v(p)=1\ \text{ and }\ v(\neg p)=0;\\ w(p)=t & \text{if} & v(p)=1\ \text{ and }\ v(\neg p)=1.\end{array}$$

This w can be homomorphically extended to the algebra $For^{0}_{\mathcal{M}}$ of $\mathcal{M}_0$-formulas. We define the translation mapping $*$ as follows:

1. $p^*=p$, if $p\in\mathcal{P}$;
2. $(\alpha\#\beta)^*=(\alpha^*\#\beta^*)$, for $\#\in\{\wedge,\vee,\rightarrow\}$;
3. $(\neg\alpha)^*=\neg_1\alpha^*$, if $v(\neg\alpha)=0$ or $v(\alpha)=v(\neg\neg\alpha)=0$;
4. $(\neg\alpha)^*=\neg_2\alpha^*$, otherwise;
5. $(\circ\alpha)^*=\circ_2\alpha^*$, if $v(\circ\alpha)=0$;
6. $(\circ\alpha)^*=\circ_1(\neg\alpha)^*$, if $v(\circ\alpha)=1$ and $v(\neg\alpha)=0$;
7. $(\circ\alpha)^*=\circ_1\alpha^*$, otherwise.

Note that the mapping $*$ is well-defined, given the definition of **mbC** (see Definition 49). The proof is now done by induction on the complexity measure of a formula α (Definition 63). Details are left to the reader. ■

THEOREM 65. [Completeness] Let $\Gamma \cup \{\alpha\}$ be a set of formulas in **mbC**. Then $\Gamma \models_{\text{PT}_0} \alpha$ implies $\Gamma \vdash_{\mathbf{mbC}} \alpha$.

Proof. Suppose that $\Gamma \models_{\text{PT}_0} \alpha$, and let v be an **mbC**-valuation such that $v(\Gamma) \subseteq \{1\}$. By Theorem 64, there is a translation $*$ and a three-valued valuation w such that, for every formula β, $w(\beta^*) \in \{T, t\}$ iff $v(\beta) = 1$. From this, $w(\Gamma^*) \subseteq \{T, t\}$ and so $w(\alpha^*) \in \{T, t\}$, because $\Gamma \models_{\text{PT}_0} \alpha$. Then $v(\alpha) = 1$. That is: For every **mbC**-valuation v, $v(\Gamma) \subseteq \{1\}$ implies $v(\alpha) = 1$. Using the completeness of **mbC** with respect to **mbC**-valuations we obtain that $\Gamma \vdash_{\mathbf{mbC}} \alpha$ as desired. ■

It is now easy to check validity for inferences in **mbC**, as shown in the following example.

EXAMPLE 66. We will prove that $\circ\alpha \vdash_{\mathbf{mbC}} \neg(\neg\alpha \wedge \alpha)$ using possible-translations semantics. We have that, for any translation $*$ in Tr_0,

$$(\circ\alpha)^* \in \{\circ_1(\alpha^*), \circ_2(\alpha^*), \circ_1\neg_1(\alpha^*), \circ_1\neg_2(\alpha^*)\},$$

$$(\neg(\neg\alpha \wedge \alpha))^* \in \{\neg_i(\neg_j(\alpha^*) \wedge \alpha^*) \ : \ i, j \in \{1, 2\}\}.$$

Let $*$ be a translation in Tr_0, w be a valuation in $\mathcal{M}_0$, and $D = \{T, t\}$. Let $x = w(\alpha^*)$, $y = w((\circ\alpha)^*)$ and $z = w((\neg(\neg\alpha \wedge \alpha))^*)$, and suppose that $y \in D$; this rules out the translation $(\circ\alpha)^* = \circ_2(\alpha^*)$ because $\circ_2(x) \notin D$. In order to prove that $z \in D$ we have the following cases:

1. $(\circ\alpha)^* = \circ_1(\alpha^*)$. Then $\circ_1(x) \in D$, thus $x \in \{T, F\}$.
   (a) $x = T$. Then $\neg_j(x) = F$ $(j \in \{1, 2\})$ and so $\neg_i(\neg_j(x) \wedge x) \in D$ for $i, j \in \{1, 2\}$.
   (b) $x = F$. Then $(\neg_j(x) \wedge x) = F$ $(j \in \{1, 2\})$ and so $\neg_i(\neg_j(x) \wedge x) \in D$ for $i, j \in \{1, 2\}$.
2. $(\circ\alpha)^* = \circ_1\neg_1(\alpha^*)$. Then $\circ_1\neg_1(x) \in D$, thus $\neg_1(x) \in \{T, F\}$ and $z = \neg_i(\neg_1(x) \wedge x)$.
   (a) $\neg_1(x) = T$. Then $x = F$ and the proof is as in (1b).
   (b) $\neg_1(x) = F$. In this case the proof is as in (1a).
3. $(\circ\alpha)^* = \circ_1\neg_2(\alpha^*)$. Then $\circ_1\neg_2(x) \in D$, thus $\neg_2(x) \in \{T, F\}$ and $z = \neg_i(\neg_2(x) \wedge x)$. From the truth-table for $\neg_2$ we obtain that $\neg_2(x) = F$, and the proof is in (1a).

This proves the desired result. On the other hand, we may prove that the converse $\neg(\neg\alpha \wedge \alpha) \vdash_{\mathbf{mbC}} \circ\alpha$ is not true in **mbC**, as announced in Theorem 41. Using the same notation as above for a given translation $*$ in Tr_0 and a valuation w in $\mathcal{M}_0$, it is enough to consider α as a propositional variable p, and choose $*$ and w such that $x = F$, and $(\circ\alpha)^* = \circ_2(\alpha^*)$. Then $z \in D$ and $y = F$. For another (counter)example, take $x = t$, $(\neg(\neg\alpha \wedge \alpha))^* = \neg_2(\neg_2(\alpha^*) \wedge \alpha^*)$ and $(\circ\alpha)^* \in \{\circ_1(\alpha)^*, \circ_1(\neg_2\alpha)^*\}$. ■

Possible-translations semantics offer an immediate decision procedure for any logic **L** that is complete with respect to a possible-translations semantical structure $\mathrm{PT} = \langle \mathcal{M}, \textsc{Tr} \rangle$ where $\mathcal{M}$ is decidable (and this is the case here, where $\mathcal{M}$ is a finite-valued logic) and $\textsc{Tr}$ is recursive. Indeed, given a formula α, if we wish to decide whether it is a theorem of **L**, it is sufficient to consider the (in this case finitely many) possible translations of α, and to check each translated formula in the corresponding three-valued matrices. Questions on the complexity of such decision procedures could be readily answered by taking into account the complexity of translations and of the three-valued matrices involved. This is a problem of independent interest, since it is immediate to see that the decision procedure of **mbC** is NP-complete: In fact, there exists a polynomial-time conservative translation from **CPL** into **mbC**, as illustrated in Theorem 71 below.

One can also use possible-translations semantics to help proving important properties about the logics in question.

REMARK 67. Recall from Remark 36 the two explosive negations given by $\wr\alpha \stackrel{\text{def}}{=} (\neg\alpha \wedge \circ\alpha)$ and $\sim\alpha \stackrel{\text{def}}{=} \alpha \to (\beta \wedge (\neg\beta \wedge \circ\beta))$, for an arbitrary β. Recall again, also, the notion of a classical negation from Definition 7. Now, while it is easy to check that $\sim$ defines a classical negation in **mbC** (the reader can, as an exercise, check that both $(\alpha \vee \sim\alpha)$ and $(\alpha \to (\sim\alpha \to \beta))$ are provable / validated by **mbC**), it is also straightforward to check that $\wr$ is not a complementing negation. Indeed, to see that α and $\wr\alpha$ can be simultaneously false, take some bottom particle $\bot = p \wedge \wr p$ and notice that $w(\bot^*) = F$, for any valuation w in $\mathcal{M}_0$ and any translation $*$ in $\textsc{Tr}_0$. Consider now some translation such that $(\circ p) = \circ_2 p$. In that case, $w((\wr\bot)^*) = F$, for any w. Then, while $\bot \models_{\mathrm{PT}_0} \wr\bot$ certainly holds good, it is not the case that $\models_{\mathrm{PT}_0} \wr\bot$. Notice moreover that, while $\wr\alpha \models_{\mathrm{PT}_0} \sim\alpha$, we have that $\sim\alpha \not\models_{\mathrm{PT}_0} \wr\alpha$. ■

We trust the above features to confirm the importance of possible-translations semantics as a philosophically apt and useful semantical tool for treating not only Logics of Formal Inconsistency but also many other logics in the literature.

*3.5 Tableau proof systems for **LFI**s*

In this section we shall use a very general method to obtain complete tableau systems for **mbC** and for C_1. The method introduced in [Caleiro *et al.*, 2004] permits one to obtain a complete tableau system for any propositional logic which has a complete semantics given through the so-called 'dyadic valuations'. Such valuations have values in $\mathbf{2} = \{0, 1\}$ (or, equivalently, in $\{T, F\}$) and are axiomatized by first-order clauses of a certain specific form.

Briefly, suppose that there is a set of axioms governing a class of valuation maps $v: For \longrightarrow \mathbf{2}$ of the form

$$(v(\varphi_1) = Q_1, \ldots, v(\varphi_n) = Q_n) \Rrightarrow (S_1 | \cdots | S_k)$$

where $n \geq 0$ and $k \geq 0$ and, for every $1 \leq i \leq k$,

$$S_i = (v(\varphi^i_1) = Q^i_1, \ldots, v(\varphi^i_{r_i}) = Q^i_{r_i}),$$

with $Q_i, Q^i_j \in \{T, F\}$ $(1 \leq j \leq r_i)$ and $r_i \geq 1$. If $n = 0$ then $(v(\varphi_1) = Q_1, \ldots, v(\varphi_n) = Q_n)$ is just $\top$. On the other hand, if $k = 0$ then $(S_1 | \cdots | S_k)$ is $\bot$. Commas ',' and bars '|' denote conjunctions and disjunctions, respectively, and '$\Rrightarrow$' denotes implication. Examples of axioms for valuations that can be put in this format are given by the **mbC**-valuations (cf. Definition 49) and by the valuation semantics of da Costa's C_1 (cf. Example 58).

For instance, clause (v5) of Definition 49 has the required form:

(v5) $v(\circ\alpha) = T \Rrightarrow (v(\alpha) = F \mid v(\neg\alpha) = F)$

whereas clause (v3) can be split in three clauses of the required form:

(v3.1) $v(\alpha \to \beta) = T \Rrightarrow (v(\alpha) = F \mid v(\beta) = T)$;
(v3.2) $v(\alpha) = F \Rrightarrow v(\alpha \to \beta) = T$;
(v3.3) $v(\beta) = T \Rrightarrow v(\alpha \to \beta) = T$.

It will be convenient in what follows to keep the more complex formulas on the left-hand side of the implication; we thus substitute (v3.2) and (v3.3) by:

(v3.4) $v(\alpha \to \beta) = F \Rrightarrow (v(\alpha) = T, v(\beta) = F)$

The next step in the algorithm described in [Caleiro *et al.*, 2004] is to 'translate' every clause of the dyadic semantics into a tableau rule by interpreting an equation '$v(\varphi) = Q$' as a signed formula $Q(\varphi)$ (recalling that $Q \in \{T, F\}$). Thus, a clause as above is transformed in a (two-signed) tableau rule of the form:

$$\begin{array}{ccc} & Q_1(\varphi_1), \ldots, Q_n(\varphi_n) & \\ \swarrow & \cdots & \searrow \\ Q^1_1(\varphi^1_1) & & Q^k_1(\varphi^k_1) \\ \vdots & & \vdots \\ Q^1_{r_1}(\varphi^1_{r_1}) & & Q^k_{r_k}(\varphi^k_{r_k}) \end{array}$$

By transforming each clause of the dyadic semantic valuation into a tableau rule, we obtain a tableau system for the given logic. In order to ensure completeness of the tableau system, it is necessary to consider two extra axioms for the valuation semantics:

(DV1) $(v(\varphi) = T,\ v(\varphi) = F) \Rrightarrow \bot$;

(DV2) $\top \Rrightarrow (v(\varphi) = T \mid v(\varphi) = F)$.

Axioms (DV1) and (DV2) guarantee that the mapping obeying them is a two-valued valuation $v : For \longrightarrow \mathbf{2}$. The translation of axiom (DV1) gives us the usual closure condition for a branch in a given tableau. On the other hand, (DV2) gives us the following branching tableau rule, R_b:

$$\frac{}{T(\varphi) \mid F(\varphi)}$$

As a consequence, the resulting tableau system loses the analytic character. Fortunately, in many important cases this branching rule can be eliminated.

Now we apply this technique to obtain a complete tableau system for logic **mbC**, based on the valuation semantics given in Definition 49.

EXAMPLE 68. We define a complete tableau system for **mbC** as follows:

$$\frac{F(\neg X)}{T(X)} \qquad \frac{T(\circ X)}{F(X) \mid F(\neg X)} \qquad \frac{}{T(X) \mid F(X)}$$

$$\frac{T(X_1 \wedge X_2)}{T(X_1),\ T(X_2)} \qquad \frac{F(X_1 \wedge X_2)}{F(X_1) \mid F(X_2)}$$

$$\frac{T(X_1 \vee X_2)}{T(X_1) \mid T(X_2)} \qquad \frac{F(X_1 \vee X_2)}{F(X_1),\ F(X_2)}$$

$$\frac{T(X_1 \to X_2)}{F(X_1) \mid T(X_2)} \qquad \frac{F(X_1 \to X_2)}{T(X_1),\ F(X_2)}$$

■

Observe that, except for the branching rule R_b, all other rules are analytic in the sense that the consequences are always less complex than the premises (recall that, as in Definition 63, $l(\circ\alpha) = l(\alpha)+2$ and $l(\neg\alpha) = l(\alpha)+1$). The results proven in [Caleiro *et al.*, 2004] guarantee that the tableau system defined above is sound and complete for **mbC**.

Another nice application of the techniques described above is the definition of a tableau system for the historical **dC**-system C_1 (see Definition 23).

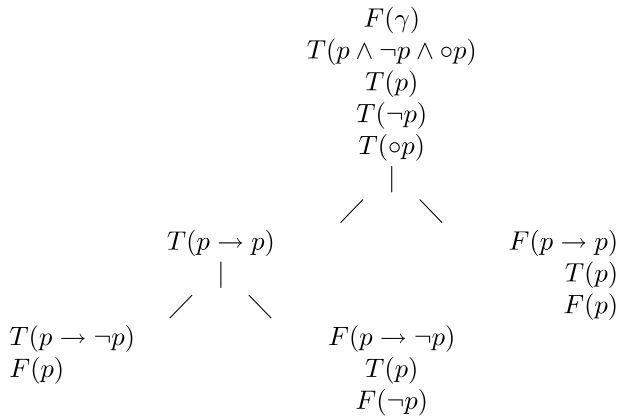
EXAMPLE 69. Recall from Example 58 the characteristic valuation semantics for the logic C_1. Those clauses of course can be formally rewritten as axioms of a dyadic semantics, using '|', '⇛' and ','. Using the above described method it is immediate to define a complete tableau system associated to those axioms. Consider indeed all the rules of the tableau system for **mbC** in Example 68 — except the rule concerning $\circ$, since it does not correspond to any axiom of a C_1-valuation — and add the following further rules:

$$\frac{T(\neg\neg X)}{T(X)} \qquad \frac{F(\circ(X_1 \# X_2))}{F(\circ X_1) \mid F(\circ X_2)} \qquad \frac{T(\circ X_2),\ T(X_1 \to X_2),\ T(X_1 \to \neg X_2)}{F(X_1)}$$

where $\# \in \{\wedge, \vee, \to\}$ and $\circ X$ abbreviates the formula $\neg(X \wedge \neg X)$. Comparing this tableau system with the one defined in [Carnielli and Lima-Marques, 1992], we see that it does not present loops. Although the looping rules proposed in that paper permit one to obtain much conciser tableau proofs, what we have here is a generic method that *automatically generates* a complete set of tableau rules (not necessarily the most convenient one). ■

It is worth reinforcing that the branching rule R_b is necessary, above, in order to obtain completeness. This rule is not strictly analytic, but can be bounded in certain cases so as to guarantee the termination of the decidable tableau procedure. Moreover, the variables occurring in the formula X must be contained in the finite collection of variables in the tableau branch.

EXAMPLE 70. Consider the tableau system for C_1 given in Example 69 and let γ be the formula $\neg(p \wedge \neg p \wedge \circ p)$, where p is a propositional variable. The formula γ is a thesis of C_1. However, it is easy to see that no C_1-tableau for the set $\{F(\gamma)\}$ can close without using the rule R_b. We show below a closed tableau for the set $\{F(\gamma)\}$ which uses R_b twice.



■

This example shows that, in general, it is not possible to eliminate R_b if we wish to obtain completeness. This holds even in case the tableau system satisfies the subformula property, as in Example 70. In certain cases R_b can be eliminated if we have, for instance, looping rules as in [Carnielli and Lima-Marques, 1992]. For the case of C_1 the tableau system treated in that paper uses the looping rule:

$$\frac{T(\neg X)}{F(X) \mid F(\circ X)},$$

while our presentation has no rule for analyzing $T(\neg X)$.

3.6 Talking about classical logic

When attempting to compare the inferential power of two logics one often finds difficulties, because those logics might not be 'talking about the same thing'. For instance, **mbC** is written in a richer signature than that of **CPL**, and so these two logics might seem hard to compare. However, it is possible to linguistically extend **CPL** by the addition of a consistency-like connective, whose matrices will be such that $\circ(x) = 1$ for every x. At the syntactic level, it suffices to add to any axiomatization of **CPL** (for convenience, let us take the one mentioned in Remark 24) the following axiom schema:

(ext) $\circ\alpha$

This (innocuous, but linguistically relevant) extension of **CPL** will here be called *extended classical logic*, or **eCPL**. Notice how we have in fact already used **eCPL** in showing how **CPL** could be seen as the minimal consistent extension of some paraconsistent logics (check, for instance, Remark 24). Clearly, despite being gently explosive, **eCPL** does not define an **LFI**, given that it is not paraconsistent. Of course, it is, in fact, a consistent logic (see Definition 4). Now, **mbC** can be characterized as a deductive fragment of **eCPL**, because all axioms of **mbC** are validated by the matrices of **eCPL**. Since **mbC** is a fragment of (an alternative formulation of) classical logic, then we can conclude that **mbC** is a non-contradictory and non-trivial logic. On the other hand, we will show in this subsection that there are several ways of encoding each inference of **CPL** within **mbC**.

First of all, recall the **DAT**s from Remark 21, the Derivability Adjustment Theorems that described how the **LFI**s could be used to recover consistent reasoning by the addition in each case of a convenient number of consistency assumptions. In particular, in logics such as **mbC**, **C**-systems based on classical logic, it should be clear how classical reasoning can be recovered. For each classical rule that is lost by paraconsistency, such as *reductio* and contraposition in items (ii) and (iii) of Theorem 30, there is an adjusted version of the same rule that is gained, as shown in Theorem 40. Indeed, it is now easy to give a semantical proof that:

$$\forall\Gamma\forall\gamma\exists\Delta(\Gamma \Vdash_{\mathbf{eCPL}} \gamma \;\text{ iff }\; \circ(\Delta), \Gamma \Vdash_{\mathbf{mbC}} \gamma).$$

Now, besides the **DAT**s, there might well be other more direct ways of recovering consistent reasoning from inside a given **LFI**. This section will show how this can be done through the use of a conservative translation, without the addition of further assumptions to the set of premisses of a given inference.

Except for negation, all other connectives of **mbC** had a classic-like behavior. The key for the next result will be, then, to make use of the classical negation $\sim$ that can be defined within **mbC** (cf. Remark 67) by setting ${\sim}\alpha \stackrel{\text{def}}{=} \alpha \to (\beta \wedge (\neg\beta \wedge \circ\beta))$, for an arbitrary β, in order to recover all classical inferences.

THEOREM 71. Let For° be the algebra of formulas for the signature Σ° of **mbC**. There is a mapping $t_1 : For \longrightarrow For^{\circ}$ that conservatively translates **CPL** inside of **mbC**, that is, for every $\Gamma \cup \{\alpha\} \subseteq For$:

$$\Gamma \vdash_{\mathbf{CPL}} \alpha \;\text{ iff }\; t_1(\Gamma) \vdash_{\mathbf{mbC}} t_1(\alpha).$$

Proof. Define the mapping t_1 recursively as follows:

1. $t_1(p) = p$, for every $p \in \mathcal{P}$;
2. $t_1(\gamma \# \delta) = t_1(\gamma) \# t_1(\delta)$, if $\# \in \{\wedge, \vee, \to\}$;
3. $t_1(\neg\gamma) = {\sim} t_1(\gamma)$.

Since both **CPL** and **mbC** are compact and have a deductive implication, and considering that t_1 preserves implications, it suffices to prove that:

$$\vdash_{\mathbf{CPL}} \alpha \;\text{ iff }\; \vdash_{\mathbf{mbC}} t_1(\alpha)$$

for every $\alpha \in For$.

That $\vdash_{\mathbf{CPL}} \alpha$ implies $\vdash_{\mathbf{mbC}} t_1(\alpha)$ is an easy consequence of the fact that $\sim$ is a classical negation within **mbC** and from the definition of the translation mapping t_1. Let's now check that $\vdash_{\mathbf{mbC}} t_1(\alpha)$ implies $\vdash_{\mathbf{CPL}} \alpha$. Consider the classical matrices for the classical connectives, and define $\circ(x) = 1$ for all x. Then $\neg\alpha$ and ${\sim}\alpha$ take the same value and so $t_1(\alpha)$ and α take the same value in this semantics. Therefore, if $t_1(\alpha)$ is a theorem of **mbC** then $t_1(\alpha)$ is valid for the above matrices and so α is valid using classical matrices. Thus, α is a theorem of **CPL**, by the completeness of classical logic. ■

In what follows, and in stronger logics than **mbC**, we will see yet other ways of recovering classical inferences inside our **LFI**s (check Theorems 93, 110 and 111).

4 A RICHER **LFI**

*4.1 The system **mCi**, and its significance*

In Remark 37 we have mentioned the possibility of defining in **mbC** an inconsistency connective that is dual to the consistency connective that is indigenous to this logic, by setting $\bullet\alpha \stackrel{\text{def}}{=} {\sim}\circ\alpha$, where ${\sim}\alpha \stackrel{\text{def}}{=} \alpha \to (\beta \wedge (\neg\beta \wedge \circ\beta))$, for an arbitrary β, is a classical negation. Now, how could we enrich **mbC** so as to be able to define the inconsistency connective instead by the direct use of the paraconsistent negation, that is, by setting $\bullet\alpha \stackrel{\text{def}}{=} \neg\circ\alpha$? This is exactly what will be done by extending **mbC** into the logic **mCi** in this subsection. In fact, this **mCi** will reveal to be a logic in which $\bullet\alpha$ and $\neg\bullet\alpha$ will be logically indistinguishable from to $\neg\circ\alpha$ and $\circ\alpha$, respectively.

From Theorem 41(i) we know that $\alpha \wedge \neg\alpha \vdash_{\mathbf{mbC}} \neg\circ\alpha$. The converse property (which does not hold in **mbC**) will be the first additional axiom we will add to **mbC** in upgrading this logic. (As we shall see, this is a weaker condition than the definition of $\circ\alpha$ as $\neg(\alpha \wedge \neg\alpha)$, as in the case of da Costa's C_1.) On the other hand, we wish that formulas of the form $\neg\circ\alpha$ 'behave classically', and we wish to obtain in fact a logic that is controllably explosive in contact with formulas of the form $\neg^n\circ\alpha$, where $\neg^0\alpha \stackrel{\text{def}}{=} \alpha$ and $\neg^{n+1}\alpha \stackrel{\text{def}}{=} \neg\neg^n\alpha$. Any formula of the form $\neg^n\circ\alpha$ would thus be assumed to 'behave classically', and $\{\neg^n\circ\alpha, \neg^{n+1}\circ\alpha\}$ would be an explosive theory. This desideratum leads us to consider the following (cf. [Marcos, 2004d]):

DEFINITION 72. The logic **mCi**, is obtained from **mbC** by the addition of the following axiom schemas:

(ci) $\neg\circ\alpha \to (\alpha \wedge \neg\alpha)$

(cc)$_n$ $\circ\neg^n\circ\alpha \qquad (n \geq 0)$

To the above axiomatization we add the definition of an inconsistency connective $\bullet$ by setting $\bullet\alpha \stackrel{\text{def}}{=} \neg\circ\alpha$. ■

Notice that $\neg\circ\alpha$ and $(\alpha \wedge \neg\alpha)$ are equivalent in **mCi**. Clearly every set $\{\neg^n\circ\alpha, \neg^{n+1}\circ\alpha\}$ is explosive in **mCi** because of (bc1) and (cc)$_n$. This expresses the 'classical behavior' of formulas of the form $\circ\alpha$ (with respect to the paraconsistent negation). That is, a formula α in general needs the extra assumption $\circ\alpha$ to 'behave classically', but the formula $\circ\alpha$ and its iterated negations always 'behave classically'. In Theorem 75 below we will see that $\neg\bullet\alpha$ is equivalent to $\circ\alpha$, and in Corollary 96, further on, such formulas will be seen in fact to be logically indistinguishable. Notice in that case how close is the bond that is established in between inconsistency and contradictoriness by way of the paraconsistent negation.

We can immediately check that the equivalence in **mCi** between $\neg\circ\alpha$ and $(\alpha \wedge \neg\alpha)$ is in fact logically weaker than the definition of $\circ\alpha$ as $\neg(\alpha \wedge \neg\alpha)$ made in C_1 (recall also Theorem 41 items (iii) and (iv)) since the latter formula implies the former, but not the other way around.

THEOREM 73. This rule hold good in **mCi**:

(i) $\neg\circ\alpha \vdash_{\mathbf{mCi}} (\alpha \wedge \neg\alpha)$,

but the following rules do not:

(ii) $\neg(\alpha \wedge \neg\alpha) \vdash_{\mathbf{mCi}} \circ\alpha$;

(iii) $\neg(\neg\alpha \wedge \alpha) \vdash_{\mathbf{mCi}} \circ\alpha$.

Proof. Item (i) is obvious. In order to prove that (ii) and (iii) do not hold in **mCi**, consider the matrices for **LFI1** (see Examples 13 and 14), taking 0 as the only non-designated value. ■

It should be clear that, even though in **mCi** there is a formula in the classical language *For* (namely, the formula $(\alpha \wedge \neg\alpha)$) which is equivalent to a formula that expresses inconsistency (the formula $\bullet\alpha$), there is no formula in the classical language which can express consistency in **mCi**. Notwithstanding, we have the following:

THEOREM 74. (i) $\neg(\alpha \wedge \neg\alpha)$ and $\neg(\neg\alpha \wedge \alpha)$ are not top particles in **mCi**.

(ii) $\circ\alpha$ and $\neg\circ\alpha$ are not bottom particles.

(iii) The schemas $(\alpha \to \neg\neg\alpha)$ and $(\neg\neg\alpha \to \alpha)$ are not provable in **mCi**.

Proof. Items (i), (ii) and the first part of item (iii) can be checked using again the matrices of $\mathbf{P}^1$, enriched with the (definable) truth-table for $\circ$ (Example 15). For the second part of item (iii) one could use for instance the tableaux of **mCi** (see Example 92). ■

It is straightforward to check the following properties of **mCi**:

THEOREM 75. The following rules hold good in **mCi**:

(i) $\neg\neg\circ\alpha \vdash_{\mathbf{mCi}} \circ\alpha$;

(ii) $\circ\alpha \vdash_{\mathbf{mCi}} \neg\neg\circ\alpha$;

(iii) $\circ\alpha, \neg\circ\alpha \vdash_{\mathbf{mCi}} \beta$;

(iv) $(\Gamma, \beta \vdash_{\mathbf{mCi}} \circ\alpha)$ and $(\Delta, \beta \vdash_{\mathbf{mCi}} \neg\circ\alpha)$ implies $(\Gamma, \Delta \vdash_{\mathbf{mCi}} \neg\beta)$.

Proof. For item (i), from $\neg\neg\circ\alpha$ and $\circ\alpha$ we obviously prove $\circ\alpha$ in **mCi**. On the other hand, from $\neg\neg\circ\alpha$ and $\neg\circ\alpha$ we also prove $\circ\alpha$ in **mCi**, because $\circ\neg\circ\alpha$ and (bc1) are axioms of **mCi**. Using proof-by-cases one concludes that $\neg\neg\circ\alpha \vdash_{\mathbf{mCi}} \circ\alpha$. The other items are proven similarly. Notice in particular how items (i) and (ii) together show that $\neg\bullet\alpha \dashv\vdash_{\mathbf{mCi}} \circ\alpha$ holds good. ■

Item (ii) of Theorem 74 and item (iii) of Theorem 75 say that **mCi** is controllably explosive in contact with $\circ p_0$ (recall Definition 8(i)). In fact, the following relation between consistency and controllable explosion holds:

THEOREM 76. Let **L** be a non-trivial extension of **mCi** with a deductive implication. A schema $\sigma(p_0, \ldots, p_n)$ is provably consistent in **L** if, and only if, **L** is controllably explosive in contact with $\sigma(p_0, \ldots, p_n)$.

Proof. If $\vdash_{\mathbf{L}} \circ\sigma(\alpha_0, \ldots, \alpha_n)$ then, by axiom (bc1),

$$\Gamma, \sigma(\alpha_0, \ldots, \alpha_n), \neg\sigma(\alpha_0, \ldots, \alpha_n) \vdash_{\mathbf{L}} \beta$$

for every Γ and every β.

Conversely, assume that $\Gamma, \sigma(\alpha_0, \ldots, \alpha_n), \neg\sigma(\alpha_0, \ldots, \alpha_n) \vdash_{\mathbf{L}} \beta$ for any Γ and β. Since $\neg{\circ}\sigma(\alpha_0, \ldots, \alpha_n) \vdash_{\mathbf{L}} (\sigma(\alpha_0, \ldots, \alpha_n) \wedge \neg\sigma(\alpha_0, \ldots, \alpha_n))$ (from (ci)) then it follows that $\neg{\circ}\sigma(\alpha_0, \ldots, \alpha_n)$ is a bottom particle. As in the proof of Theorem 38(i) (using here the fact that **L** has a deductive implication), we get $\vdash_{\mathbf{L}} \neg\neg{\circ}\sigma(\alpha_0, \ldots, \alpha_n)$. By Theorem 75(i), we conclude $\vdash_{\mathbf{L}} {\circ}\sigma(\alpha_0, \ldots, \alpha_n)$. ■

Complementing the properties of contraposition treated in Theorem 40, we have:

THEOREM 77. These are some restricted forms of contraposition introduced by **mCi**:
(i) $(\alpha \to {\circ}\beta) \vdash_{\mathbf{mCi}} (\neg{\circ}\beta \to \neg\alpha)$;
(ii) $(\alpha \to \neg{\circ}\beta) \vdash_{\mathbf{mCi}} ({\circ}\beta \to \neg\alpha)$;
(iii) $(\neg\alpha \to {\circ}\beta) \vdash_{\mathbf{mCi}} (\neg{\circ}\beta \to \alpha)$;
(iv) $(\neg\alpha \to \neg{\circ}\beta) \vdash_{\mathbf{mCi}} ({\circ}\beta \to \alpha)$.

Proof. (i) By axiom $(\text{cc})_0$, ${\circ\circ}\beta$ is a theorem of **mCi**. The result now follows from Theorem 40(iii). The other items are proven similarly. ■

On the other hand, properties such as $({\circ}\alpha \to \beta) \vdash_{\mathbf{mCi}} (\neg\beta \to \neg{\circ}\alpha)$ do not hold; this can easily be checked after Corollary 89 to be established below. Notice how the above theorem opens yet another way for the internalization of classical inferences, as discussed in Subsection 3.6.

Recall now the replacement property (RP) discussed in Remark 43. We had already proven in Theorem 44 that (RP) cannot hold in certain paraconsistent extensions of **mbC**. On what concerns its possible validity in paraconsistent extensions of **mCi**, we can prove that:

THEOREM 78.
(i) The replacement property (RP) is not enjoyed by **mCi**.
The replacement property (RP) cannot hold in any paraconsistent extension of **mCi** in which:
(ii) $\neg(\neg\alpha \wedge \neg\beta) \vdash_{\mathbf{mbC}} (\alpha \vee \beta)$ holds; or
(iii) $(\neg\alpha \vee \neg\beta) \vdash_{\mathbf{mbC}} \neg(\alpha \wedge \beta)$ holds.

Proof. (i) Consider again the first set of matrices (with the same designated values) used in the proof of Theorem 42.
(ii) Consider the supplementing negation ${\wr}\,\alpha = (\neg\alpha \wedge {\circ}\alpha)$ for **mCi** proposed in Remark 36. By Theorem 75 this last formula is equivalent to $(\neg\alpha \wedge \neg\neg{\circ}\alpha)$. In Theorem 90, this negation will be shown to behave classically inside this logic. But then, $\neg{\wr}\alpha \vdash \alpha \vee \neg{\circ}\alpha$, by hypothesis, and so $\neg{\wr}\alpha \vdash \alpha$, using axiom (ci), proof-by-cases and conjunction elimination. The rest of the proof now follows exactly like in Theorem 44(ii).
Finally, for item (iii), recall that, from (Ax10), $(\neg\alpha \vee \neg\neg\alpha)$ is a theorem of **mCi**. But then, by hypothesis, $\neg(\alpha \wedge \neg\alpha)$ would also be a theorem. From

Theorem 41(ii) and replacement it follows that $\neg\neg\circ\alpha$ is provable, and by Theorem 75(i) this results in $\circ\alpha$ being provable. Thus, the resulting logic will be explosive. ■

In the case of the logic **mbC**, we have called the attention of the reader to the fact that the validity of (RP) required the validity of rules (EC) and (EO) (see the end of Subsection 3.2). Curiously, now in **mCi** we can check that (EC) is enough:

THEOREM 79. In extensions of **mCi** the validity of:

(EC) $\forall\alpha\forall\beta((\alpha \dashv\Vdash \beta)$ implies $(\neg\alpha \dashv\Vdash \neg\beta))$

guarantees the validity of:

(EO) $\forall\alpha\forall\beta((\alpha \dashv\Vdash \beta)$ implies $(\circ\alpha \dashv\Vdash \circ\beta))$.

Proof. Suppose $(\alpha \dashv\vdash \beta)$. By (EC) we have that $(\neg\alpha \dashv\vdash \neg\beta)$, and from these two equivalences we conclude that $(\alpha \wedge \neg\alpha) \dashv\vdash (\beta \wedge \neg\beta)$. But from Theorems 41(ii) and 73(i) we have that $\neg\circ\gamma \dashv\vdash_{\mathbf{mCi}} (\gamma \wedge \neg\gamma)$, so we have that $\neg\circ\alpha \dashv\vdash \neg\circ\beta$. By Theorem 77(iv) we conclude then that $\circ\alpha \dashv\vdash \circ\beta$. ■

Suppose now we considered the addition to **mCi** of a stronger rule than (EC), in order to ensure replacement:

THEOREM 80. Consider the following rule:

(RC) $\forall\alpha\forall\beta((\alpha \Vdash \beta)$ implies $(\neg\beta \Vdash \neg\alpha))$.

The least extension **L** of **mCi** which satisfies (RC) and proof-by-cases collapses into classical logic.

Proof. From the axioms of **mCi** we first obtain $\neg\circ\alpha \vdash_{\mathbf{L}} \alpha$, and $\neg\circ\alpha \vdash_{\mathbf{L}} \neg\alpha$. By (RC) and Theorem 75(i) we then get $\neg\alpha \vdash_{\mathbf{L}} \circ\alpha$ and $\neg\neg\alpha \vdash_{\mathbf{L}} \circ\alpha$. But then, using the proof-by-cases, we conclude that $\vdash_{\mathbf{L}} \circ\alpha$, that is, all formulas are consistent. The result now follows, as usual, from Remark 24. ■

Notice that our paraconsistent formulation of the normal modal logics from Example 26 do *not* extend the logic **mCi** (contrast this with Remark 45 about **mbC**). As we said at the beginning of this subsection, an inconsistency connective $\bullet$ can often be defined from a consistency connective $\circ$ by taking $\sim\circ$, where $\sim$ is a classical negation. The definition of an inconsistency connective by taking $\neg\circ$ is an innovation of **mCi** over **mbC**, and it is typical in fact of most **LFI**s from the literature, as the ones we will be studying in the rest of this chapter. The reader should not think though that the latter class of **C**-systems has any intrinsic advantage over the former. This far, it only seems to have more often met the intuitions of the authors, for some reason or another — or maybe by pure chance. In reality, the distinction between the two classes is only clear in a framework such as the one set in the present study, where consistency and inconsistency are taken as (primitive or defined) connectives in their own right.

4.2 Other features of ***mCi***

In this subsection we will extend to **mCi** the results obtained for **mbC** in Subsections 3.3, 3.4, 3.5 and 3.6. That is, we will introduce a valuation semantics, a possible-translations semantics and a tableau system for **mCi**. Finally, we will exhibit some conservative translations from classical logic into **mCi**.

We begin with a brief description of a valuation semantics for **mCi**, in the same manner as it was done in Subsection 3.3 with **mbC**. The plan of action is similar to that of **mbC**, and we just outline the main points. First of all, observe that **mCi** is not a finite-valued logic, as a direct consequence of Theorems 46 or 47.

THEOREM 81. The logic **mCi** is not characterizable by finite matrices. ■

DEFINITION 82. An **mCi**-valuation is an **mbC**-valuation $v : For^\circ \longrightarrow \mathbf{2}$ (see Definition 49) satisfying, additionally, the following:

(v6) $v(\neg\circ\alpha) = 1$ implies $v(\alpha) = 1$ and $v(\neg\alpha) = 1$;
(v7.n) $v(\circ\neg^n\circ\alpha) = 1$ (for $n \geq 0$). ■

The semantic consequence relation obtained from **mCi**-valuations will be denoted by $\vDash_{\mathbf{mCi}}$. It is easy to prove soundness for **mCi** with respect to **mCi**-valuations.

THEOREM 83. [Soundness] Let $\Gamma\cup\{\alpha\}$ be a set of formulas in For°. Then: $\Gamma \vdash_{\mathbf{mCi}} \alpha$ implies $\Gamma \vDash_{\mathbf{mCi}} \alpha$. ■

The proof of completeness is similar to that of **mbC**, but obviously changing $\vdash_{\mathbf{mbC}}$ by $\vdash_{\mathbf{mCi}}$: Given a set of formulas $\Delta\cup\{\alpha\}$ in For° we say that Δ is a *relatively maximal with respect to* α *in* **mCi** if $\Delta \nvdash_{\mathbf{mCi}} \alpha$ and for any formula β in For° such that $\beta \notin \Delta$ we have $\Delta, \beta \vdash_{\mathbf{mCi}} \alpha$. As in Lemma 53, relatively maximal theories are maximal. An analogue to Lemma 54 can then be proven:

LEMMA 84. Let $\Delta \cup \{\alpha\}$ be a set of formulas in For° such that Δ is relatively maximal with respect to α in **mCi**. Then Δ satisfies the properties (i)–(v) of Lemma 54, plus the following:

(vi) $\neg\circ\beta \in \Delta$ implies $\beta \in \Delta$ and $\neg\beta \in \Delta$.

(vii) $\circ\neg^n\circ\beta \in \Delta$. ■

COROLLARY 85. The characteristic function of a relatively maximal theory of **mCi** defines an **mCi**-valuation. ■

THEOREM 86. [Completeness w.r.t. valuation semantics] Let $\Gamma \cup \{\alpha\}$ be a set of formulas in For°. Then: $\Gamma \vDash_{\mathbf{mCi}} \alpha$ implies $\Gamma \vdash_{\mathbf{mCi}} \alpha$. ■

Next, as it was done in Subsection 3.4 with the logic **mbC**, we can also provide an alternative semantics for **mCi** in terms of possible-translations semantics.

Consider the collection $\mathcal{M}_1$ of three-valued matrices formed by the matrices introduced in Subsection 3.4, but now considering just one consistency operator called $\circ_3$ instead of $\circ_1$ and $\circ_2$, presented by the truth-table:

	$\circ_3$
T	T
t	F
F	T

Again, T and t are the designated values. In $\mathcal{M}_1$, the only truth-value that is not consistent is t. If $For^1_{\mathcal{M}}$ denotes the algebra of formulas generated by $\mathcal{P}$ over the signature of $\mathcal{M}_1$, let's consider the set $\text{T}\textsc{r}_1$ of all functions $*: For^\circ \longrightarrow For^1_{\mathcal{M}}$ obeying the clauses $(tr0)$–$(tr2)$ of translations introduced in Subsection 3.4, plus the following:

$(tr3)_1$ $\quad (\circ\alpha)^* \in \{\circ_3\alpha^*, \circ_3(\neg\alpha)^*\}$;

$(tr4)_1$ $\quad (\neg^{n+1}\circ\alpha)^* = \neg_1(\neg^n\circ\alpha)^*$.

We say the pair $\text{PT}_1 = \langle \mathcal{M}_1, \text{T}\textsc{r}_1 \rangle$ is a *possible-translations semantical structure for* **mCi**. If $\vDash^1_{\mathcal{M}}$ denotes the consequence relation in $\mathcal{M}_1$, and $\Gamma \cup \{\alpha\}$ is a set of formulas of **mCi**, the PT_1-consequence relation, $\models_{\text{PT}_1}$, is defined as:

$$\Gamma \models_{\text{PT}_1} \alpha \quad \text{iff} \quad \Gamma^* \vDash^1_{\mathcal{M}} \alpha^* \quad \text{for all } * \in \text{T}\textsc{r}_1.$$

We leave to the reader the proof of the following easy result:

THEOREM 87. [Soundness] Let $\Gamma \cup \{\alpha\}$ be a set of formulas of **mCi**. Then $\Gamma \vdash_{\textbf{mCi}} \alpha$ implies $\Gamma \models_{\text{PT}_1} \alpha$. ■

The proof of completeness follows the same lines than the one obtained for **mbC** (see [Marcos, 2004d]).

THEOREM 88. [Representability] Given an **mCi**-valuation v there is a translation $*$ in $\text{T}\textsc{r}_1$ and a valuation w in $\mathcal{M}_1$ such that, for every formula α in **mCi**:

$$w(\alpha^*) = T \quad \text{implies} \quad v(\neg\alpha) = 0; \quad \text{and}$$

$$w(\alpha^*) = F \quad \text{iff} \quad v(\alpha) = 0.$$

Proof. The proof is similar to that of Theorem 64, but now defining $(\circ\alpha)^* = \circ_3(\neg\alpha)^*$ if $v(\neg\alpha) = 0$, and $(\circ\alpha)^* = \circ_3\alpha^*$ otherwise. Finally, set $(\neg^{n+1}\circ\alpha)^* = \neg_1(\neg^n\circ\alpha)^*$. Details are left to the reader. ■

COROLLARY 89. [Completeness w.r.t possible-translations semantics] Let $\Gamma \cup \{\alpha\}$ be a set of formulas in **mCi**. Then $\Gamma \models_{\text{PT}_1} \alpha$ implies $\Gamma \vdash_{\textbf{mCi}} \alpha$.

In Remark 36 we have defined two supplementing negations for **mbC**, $\wr$ and $\sim$, and in Remark 67 we have shown that only one of them, namely $\sim$, was classical in **mbC**. Now we can use the possible-translations semantics of **mCi** to check that in this logic the two negations produce logically indistinguishable formulas:

THEOREM 90. Given a formula α, the formulas $\wr\alpha$ and $\sim\alpha$ are logically indistinguishable in **mCi**. As a result, $\wr$ defines a classical negation in **mCi**.

Proof. Recall the definition of logical indistinguishableness from Remark 43. Notice that, using the above possible-translations semantics for **mCi**, the formulas $\wr p$ and $\sim p$ produce exactly the same matrices. ■

Now, using the general techniques introduced in [Caleiro *et al.*, 2004] we can easily obtain an adequate tableau system for **mCi**, in the same way that was done for **mbC** in Subsection 3.5. Thus, in view of the valuation semantics for **mCi** stated in Definition 82 from the valuation semantics for **mbC**, it is enough to define the following:

DEFINITION 91. We define a tableau system for **mCi** by adding to the tableau system for **mbC** introduced in Example 68 the following rules:

$$\frac{T(\neg\circ X)}{T(X),\ T(\neg X)} \qquad\qquad \frac{}{T(\circ\,\neg^n\circ\,X)} \quad (\text{for } n \geq 0)$$

■

EXAMPLE 92. The above tableau system for **mCi** can be used to show that $\neg\neg p \to p$ is not a thesis of this logic. Indeed, notice that the following is an open tableau for this formula:

$F(\neg\neg p \to p)$
$T(\neg\neg p)$
$F(p)$
$T(\circ\circ p)$

Left branch: $F(\circ p)$

Right branch: $F(\neg\circ p)$, $T(\circ p)$, which branches into:
- $F(p)$
- $T(\neg p)$

None of the branches are closed: Recall indeed that we have no primitive rules for analyzing $T(\neg X)$ or $F(\circ X)$. It is easy to see that further applications of the branching rule R_b, or of the new rules of **mCi**, will not help in closing the tableau this time. ■

Finally, let's talk again about classical logic. In Theorem 71 of Subsection 3.6 we have seen how **CPL** can be encoded inside **mbC** through a conservative translation. Clearly, that same translation works for **mCi**. We will now show how it is possible to encode **eCPL** inside **mCi**, in a similar fashion.

THEOREM 93. Let For° be the algebra of formulas for the signature Σ° of **mCi**. Consider any mapping $t_2 : For^\circ \longrightarrow For^\circ$ such that:

1. $t_2(p) = p$, for every $p \in \mathcal{P}$;
2. $t_2(\gamma\#\delta) = t_2(\gamma)\#t_2(\delta)$, if $\# \in \{\wedge, \vee, \rightarrow\}$;
3. $t_2(\neg\gamma) \in \{\sim t_2(\gamma), \wr t_2(\gamma)\}$;
4. $t_2(\circ\gamma) = \circ\circ t_2(\gamma)$.

Then, t_2 is a conservative translation from **eCPL** into **mCi**.

Proof. The proof is almost identical to that of Theorem 71. The only novel clause is 4, but it is clear how it works (recall axiom $(\text{cc})_0$). ■

4.3 Inconsistency operator as primitive

Up to now we have concentrated almost exclusively on the formal notion of consistency; formal inconsistency has appeared only derivatively, defined with the help of a classical or of a paraconsistent negation. It is equally natural, however, to provide alternative axiomatizations for the logic **mCi** starting from a primitive inconsistency connective. We will now show how to do that in two different ways, one in terms of $\circ$ and $\bullet$, and the other in terms of $\bullet$ alone.

Let $\Sigma^\bullet$ and $\Sigma^{\circ\bullet}$ be the extensions of the signature Σ (recall Remark 12) obtained by the addition of a new unary connective $\bullet$ and two unary connectives $\circ$ and $\bullet$, respectively. Let $For^\bullet$ and $For^{\circ\bullet}$ be the respective algebras of formulas.

THEOREM 94. Here are two possible alternative axiomatizations for **mCi**: (1) One in $For^{\circ\bullet}$, given by the axioms (ci) and $(\text{cc})_n$ from the axiomatization of **mCi** (see Definition 72), plus the following axiom schemas:

(bc2) $\neg\bullet\alpha \rightarrow \circ\alpha$

(eq1) $\bullet\alpha \leftrightarrow \neg\circ\alpha$

This axiomatization in the 'full language' contains no abbreviations.

(2) Another one in $For^\bullet$, given by axiom schemas (Ax1)–(Ax10) and inference rule (MP), plus the following axiom schemas:

(bc1)' $\neg\bullet\alpha \rightarrow (\alpha \rightarrow (\neg\alpha \rightarrow \beta))$

(ci)’ $\bullet\alpha \to (\alpha \wedge \neg\alpha)$

(cc)’$_n$ $\neg\bullet\neg^n\bullet\alpha \qquad (n \geq 0)$

In this axiomatization, $\circ$ can be introduced by abbreviation: $\circ\alpha \stackrel{\text{def}}{=} \neg\bullet\alpha$. ■

The corresponding valuation semantics, possible-translations semantics and tableau procedures for the versions of **mCi** in the above signatures can be easily implemented and we will not annoy the reader with details.

Considering the signature $\Sigma^{\circ\bullet}$, notice that the following clauses are valid in an **mCi**-valuation:

(w1) $v(\neg\bullet\alpha) = 1$ implies $v(\circ\alpha) = 1$;
(w2) $v(\bullet\alpha) = 1$ iff $v(\neg\circ\alpha) = 1$;
(w3) $v(\bullet\alpha) = 1$ implies $v(\alpha) = 1$ and $v(\neg\alpha) = 1$;
(w4) $v(\neg\bullet\alpha) = 1$ implies $v(\alpha) = 0$ or $v(\neg\alpha) = 0$.

As a consequence of this, we can prove the following nice properties in and about **mCi**:

THEOREM 95. Fix the signature $\Sigma^{\circ\bullet}$. Let $\varphi(p)$ be a formula depending on a single propositional variable p and $\psi[p]$ a formula in which the variable p occurs (recall Remark 43). Let α be a formula, v an **mCi**-valuation over $For^{\circ\bullet}$ and $\boxplus \in \{\circ, \bullet, \neg\circ, \neg\bullet\}$. Then:

1. $v(\circ\varphi(\boxplus\alpha)) = 1$;
2. $v(\bullet\varphi(\boxplus\alpha)) = 0$;
3. $v(\psi[p/\circ\alpha]) = v(\psi[p/\neg\bullet\alpha])$;
4. $v(\psi[p/\bullet\alpha]) = v(\psi[p/\neg\circ\alpha])$. ■

COROLLARY 96. If $\varphi(p)$ is a formula depending on a single propositional variable p, α a formula and $\boxplus \in \{\circ, \bullet, \neg\circ, \neg\bullet\}$, then:

1. $\varphi(\boxplus\alpha), \neg\varphi(\boxplus\alpha) \vdash_{\mathbf{mCi}} \beta$, for every $\beta \in For^{\circ\bullet}$;
2. $\vdash_{\mathbf{mCi}} \circ\varphi(\boxplus\alpha)$;
3. $\nvdash_{\mathbf{mCi}} \bullet\varphi(\boxplus\alpha)$;
4. the formulas $\circ\alpha$ and $\neg\bullet\alpha$ are logically indistinguishable in **mCi**;
5. the formulas $\bullet\alpha$ and $\neg\circ\alpha$ are logically indistinguishable in **mCi**. ■

Obviously, as a consequence of the two last items, pairs of formulas such as $\circ\alpha$ and $\neg\neg\circ\alpha$, or such as $\bullet\alpha$ and $\neg\neg\bullet\alpha$, will also turn out to be logically indistinguishable in **mCi**. That strengthens results from Theorem 75.

*4.4 Enhancing **mCi** by dealing with double negations*

In this subsection we will see what happens when the logics **mbC** and its extension **mCi** are extended with axioms dealing with double negations, namely:

(cf) $\neg\neg\alpha \to \alpha$

(ce) $\alpha \to \neg\neg\alpha$

From item (iii) of Theorem 74 we know that neither (ce) nor (cf) is provable in **mCi**. Adding such axioms makes the negation of this logic a bit closer to classical negation. Moreover, we will see that adding them might help in simplifying the axiomatic presentations of the resulting logics, and have a consequence in the interaction of negation with the connectives for consistency and inconsistency. Note that (cf) has already appeared as (Ax11) in Definition 23.

DEFINITION 97. Consider the signature Σ°. Recall the axiomatizations of **mbC** and **mCi** from Definitions 34 and 72. Then:

1. **bC** is axiomatized as **mbC** plus (cf).
2. **Ci** is axiomatized as **mCi** plus (cf).
3. **mbCe** is axiomatized as **mbC** plus (ce).
4. **mCie** is axiomatized as **mCi** plus (ce).
5. **bCe** is axiomatized as **bC** plus (ce).
6. **Cie** is axiomatized as **Ci** plus (ce). ■

It is easy to check that:

THEOREM 98.
(i) $\circ\alpha \vdash_{\mathbf{Ci}} \circ\neg\alpha$;
(ii) $\bullet\neg\alpha \vdash_{\mathbf{Ci}} \bullet\alpha$;
(iii) $\circ\neg\alpha \vdash_{\mathbf{mCie}} \circ\alpha$;
(iv) $\bullet\alpha \vdash_{\mathbf{mCie}} \bullet\neg\alpha$. ■

Using this result one might provide a simpler and finitary axiomatization for the logic **Ci** (thus also for **Cie**):

THEOREM 99. The logic **Ci** can be obtained from **mbC** by adding the axiom schemas (ci) (cf. Definition 72) and (cf):

(ci) $\neg\circ\alpha \to (\alpha \wedge \neg\alpha)$

(cf) $\neg\neg\alpha \to \alpha$ ■

As regards semantical interpretations, notice that:

THEOREM 100. The logics from Definition 97 are not characterizable by finite matrices.

Proof. To check that for **bCe** and **Cie**, make use of Theorem 46 again. For the other logics, use either the same theorem or Theorem 47. ■

It is straightforward to endow these new systems with adequate valuation semantics, using the methods from previous sections. Indeed:

THEOREM 101. Axiom (cf) corresponds to the following clause on the definition of a valuation semantics:

(v8) $v(\neg\neg\alpha) = 1$ implies $v(\alpha) = 1$.

Similarly, axiom (ce) corresponds to:

(v9) $v(\alpha) = 1$ implies $v(\neg\neg\alpha) = 1$. ■

We can also obtain adequate tableaux for these systems, as in previous sections. Possible-translations semantics for **bC**, **Ci**, **bCe** and **Cie** can be found in [Marcos, 2004d]. These four logics were exhaustively studied in [Carnielli and Marcos, 2002].

5 ADDITIONAL TOPICS ON **LFI**S

*5.1 The **dC**-systems*

As we have seen in Corollary 96, extensions of the logic **mCi**, introduced in Definition 72, are such that the formulas $\bullet\alpha$ and $\neg\circ\alpha$ are logically indistinguishable. Moreover, we also know that the formulas $\bullet\alpha$ and $(\alpha \wedge \neg\alpha)$ are equivalent in **mCi**. But the latter are not logically indistinguishable. Indeed, as we know from Theorem 73, the formulas $\neg\bullet\alpha$ and $\neg(\alpha \wedge \neg\alpha)$ are not equivalent, neither are the formulas $\neg\neg\bullet\alpha$ and $\neg\neg(\alpha \wedge \neg\alpha)$, and so on. It seems only natural, thus, to consider extensions of **mCi** in which these last equivalences are strengthened into logical indistinguishability. This maneuver will lead us, of course, to the class of **LFI**s known as **dC**-systems, in which the new connectives of consistency and inconsistency can be dismissed from the start, and introduced by definitions written in terms of the other connectives of the language.

DEFINITION 102. The logic **mCil**, defined over the signature Σ°, is obtained from **mCi** by the addition of the following axiom schema:

(cl) $\neg(\alpha \wedge \neg\alpha) \to \circ\alpha$.

Other logics can be obtained in a similar way, such as the logics **Cil**, **mCile**, or **Cile**, obtained respectively by the addition of (cf), (ce), or both (cf) and (ce) to **mCil** (recall Subsection 4.4). ■

Obviously:

THEOREM 103. There is no paraconsistent extension of **mCil** in which the schema $\neg(\alpha \wedge \neg\alpha)$ is provable. ■

There are, however, other paraconsistent extensions of **Ci**, such as **LFI1** (see Example 14 and Theorem 119) or all non-degenerate normal modal logics extending the system T (see Example 26), in which the above schema *is* valid.

It can be checked that **mCil** is indeed a **dC**-system based on classical logic. Indeed:

THEOREM 104. The logic **mCil** can be defined over Σ using the following abbreviations: $\bullet\alpha \stackrel{\text{def}}{=} (\alpha \wedge \neg\alpha)$ and $\circ\alpha \stackrel{\text{def}}{=} \neg(\alpha \wedge \neg\alpha)$.

Proof. It can immediately be seen, from the above definitions, that the axioms (bc1), (ci) and (cl) will still hold if one just substitutes all occurrences of the operators $\bullet$ and $\circ$ by their new definitions. ■

The schema $\neg(\alpha \wedge \neg\alpha)$ played an important role in the original construction of the logics C_n (recall Definition 23), and it has often been identified with the so-called 'Principle of Non-Contradiction'. Notice, however, that such an identification is not possible with our present definition of this principle (Principle (1) in Subsection 2.1).

There is no consensus in the literature on what concerns the status of the schema $\neg(\alpha \wedge \neg\alpha)$ inside paraconsistent logics. Its validity has been criticized by some (see, for instance, [Béziau, 2002a]). A good technical reason for expecting this schema to fail is connected to the possible consequent failure of the replacement property, as predicted in Theorem 44(iv). On the other hand, the proposal of paraconsistent logics in which this schema does not hold has *also* been criticized, as for instance in [Routley and Meyer, 1976], where it is claimed that, for dialectical logics (i.e., for logics disrespecting our version of the Principle of Non-Contradiction), not only we usually have that $\neg(\alpha \wedge \neg\alpha)$ is a theorem, but that also does not conflict with other logical truths of such logics. On our approach, this whole controversy seems artificial and ill-advised. It might well be just a sterile offspring of the misidentification of the Principle of Explosion and the Principle of Non-Contradiction: In general, only the former should worry a paraconsistent logician, the latter being a much less demanding and a very often observed principle (check the ensuing discussion in section 3.8 of [Carnielli and Marcos, 2002]).

As a consequence of Corollary 96 and axioms $(\text{cc})_n$, the logic **mCil** can prove theorems of the form $\circ(\alpha\wedge\neg\alpha)$ and $\circ\neg(\alpha\wedge\neg\alpha)$. This would also raise protests by some authors (see for instance [Sylvan, 1990]).

Using again the matrices from Theorem 46 or from Theorem 47, one can immediately check that:

THEOREM 105.
The logic **mCil** is not characterizable by finite matrices. ■

Of course, we can obtain a valuation semantics for **mCil** by considering maps $v : For \longrightarrow \mathbf{2}$ satisfying axioms (v1)–(v5) of Definition 49, plus the following:

THEOREM 106. Axiom (cl) corresponds to the following clause on the definition of a valuation semantics:

(v10) $v(\neg(\alpha \wedge \neg\alpha)) = 1$ implies $v(\alpha) = 0$ or $v(\neg\alpha) = 0$. ■

At this point it should be obvious to the reader how the tableaux for this logic would look like.

If the reader has still not gotten used to the failure of the replacement property, he might be surprised with the following asymmetry allowed by the logic **mCil**:

THEOREM 107. Despite the possibility of expressing the consistency operator $\circ\alpha$, in **mCil**, by the formula $\neg(\alpha \wedge \neg\alpha)$, this operator cannot be expressed alternatively by the formula $\neg(\neg\alpha \wedge \alpha)$. In fact, while it is not possible to add the formula $\neg(\alpha \wedge \neg\alpha)$ to **mCil** without losing its paraconsistent character, the logic resulting from the addition of $\neg(\neg\alpha \wedge \alpha)$ to **mCil** is paraconsistent.

Proof. The first set of matrices from the proof of Theorem 42 show that there are atomic formulas p and q such that $\neg(\neg p \wedge p), p, \neg p$ take designated values, while q does not: Just assign $\frac{1}{2}$ to p and 0 to q. The same matrices provide a model of **mCil** plus $\neg(\neg\alpha \wedge \alpha)$, and they define a paraconsistent logic. ■

The above asymmetry has been pointed out in Theorem 4 of [Urbas, 1989] for the case of the logic C_1, an extension of **mCil** (see Remark 109). It has remained hidden for a long time within the realm of the logics C_n. Indeed, the first decision procedure offered for the logic C_1 in terms of quasi matrices, in [da Costa and Alves, 1977], was mistaken exactly in assuming $\neg(\alpha \wedge \neg\alpha)$ and $\neg(\neg\alpha \wedge \alpha)$ to be equivalent formulas.

Some natural alternatives to (cl) can immediately be considered:

(cd) $\neg(\neg\alpha \wedge \alpha) \rightarrow \circ\alpha$;

(cb) $(\neg(\alpha \wedge \neg\alpha) \vee \neg(\neg\alpha \wedge \alpha)) \rightarrow \circ\alpha$.

(RG) $\beta \dashv\vdash \alpha \wedge \neg\alpha$ implies $\neg\beta \dashv\vdash \neg(\alpha \wedge \neg\alpha)$

Clearly, the addition to **mCi** of the axiom (cd) instead of the axiom (cl), would produce a logic in which the asymmetry pointed out in Theorem 107 is inverted. That problem can be cured if the axiom (cb) is added instead, as that move produces a logic in which both $\neg(\alpha \wedge \neg\alpha)$ and $\neg(\neg\alpha \wedge \alpha)$

express consistency. However, that will not make the difficulties about the replacement property, (RP), go away. In fact, the equivalence of similar more complex formulas would not be guaranteed by (cb): It can be shown for instance that the formulas $\neg(\alpha \wedge (\alpha \wedge \neg\alpha))$ and $\neg((\alpha \wedge \neg\alpha) \wedge \alpha)$ are not automatically equivalent, even though $(\alpha \wedge (\alpha \wedge \neg\alpha))$ and $((\alpha \wedge \neg\alpha) \wedge \alpha)$ are equivalent on any **C**-system based on classical logic. As pointed out in [Carnielli and Marcos, 2002], a way of solving that problem without necessarily going as far as establishing the validity of (RP) is simply by adding the rule (RG). Of course, **dC**-systems with full (RP) are available, as it was illustrated by the modal logics proposed in Example 26 to extend the logic **mbC** (recall Remark 45).

It should be clear that each **dC**-system can in principle generate an infinite number of other **dC**-systems, if one applies the same strategy as that of the C_n logics, for $1 \leq n < \omega$ (cf. Definition 23), namely, if one simply requires stronger and stronger conditions to be met in order to establish the consistency of a formula.

5.2 Adding modularity: Letting consistency propagate

Given a class of consistent formulas, an important issue is to understand how this consistency propagates towards simpler or more complex formulas. As we have seen in Theorem 98, the addition to **mCi** of axioms or rules controlling the behavior of double negated formulas reflects directly on the propagation of negation through negation. As we will see in this subsection, one can in fact produce interesting variations on the recipe that constructs **LFI**s by directly controlling the way consistency propagates.

DEFINITION 108.
(i) The logic **Cia** is obtained by the addition of the following axiom schemas (recall Definition 23) to **Ci** (see Definition 97):

(ca1) $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \wedge \beta)$;

(ca2) $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \vee \beta)$;

(ca3) $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \to \beta)$.

(ii) The logic **Cila** is obtained by the addition of the axiom schema (cl) to **Cia** (see Definition 102). Using axioms (cf) or (cb) instead of (cl) one might similarly define the logics **Cifa**, **Cida** or **Ciba**. Adding axiom (ce) to those systems one might define the logics **Cilae**, **Cifae**, **Cidae** and **Cibae**. ■

REMARK 109. It is worth insisting that the only difference between **Cila** and the original formulation of C_1 is that the connective $\circ$ in C_1 was not taken as primitive, but $\circ\alpha$, originally denoted as α°, was assumed from the start to be an abbreviation of the formula $\neg(\alpha \wedge \neg\alpha)$. For the other logics in the hierarchy C_n, $1 \leq n < \omega$, the formula $\circ\alpha$ abbreviates more and more complex formulas, or sets of formulas, as it can be seen in Definition 23. ■

Taking into account the above definition, it is easy to prove in **Cia** the following particular version of Derivability Adjustment Theorem (recall Remark 21 and compare with what was said at the beginning of Subsection 3.6):

THEOREM 110.
Let Π denote the set of atomic formulas occurring in $\Gamma \cup \{\alpha\}$.
Then, $\Gamma \vdash_{\mathbf{CPL}} \alpha$ iff there is some $\Delta \subseteq \Pi$ such that $\circ(\Delta), \Gamma \vdash_{\mathbf{Cia}} \alpha$. ■

As pointed out already in [da Costa, 1993] and [da Costa, 1974], the same result holds in any logic C_n, taking in each case the appropriate definition of $\circ\alpha$.

Recalling that **eCPL** is just the classical propositional logic **CPL** plus the axiom schema $\circ\alpha$, we also obtain the following alternative way of recovering classical reasoning inside our present **LFI**s:

THEOREM 111. Consider the mapping $t_3 : \ For^{\circ} \longrightarrow For^{\circ}$ defined recursively as follows:

1. $t_3(p) = \circ p$, for every $p \in \mathcal{P}$;
2. $t_3(\gamma \# \delta) = (t_3(\gamma) \# t_3(\delta))$, if $\# \in \{\wedge, \vee, \rightarrow\}$;
3. $t_3(\#\gamma) = \# t_3(\gamma)$, if $\# \in \{\neg, \circ\}$.

Then t_3 conservatively translates **eCPL** inside of **Cia**.

Proof. It is enough to prove that $\vdash_{\mathbf{eCPL}} \alpha$ iff $\vdash_{\mathbf{Cia}} t_3(\alpha)$.

We first prove from left to right. Given a formula $\alpha(p_1, \ldots, p_n)$ in For°, then $t_3(\alpha) = \alpha(\circ p_1, \ldots, \circ p_n)$. From this, using axioms (ca1)–(ca3), axiom $(\mathrm{cc})_n$ (Definition 72) and Theorem 98(i) it is not difficult to prove by induction on the complexity $l(\alpha)$ of α that $\vdash_{\mathbf{Cia}} \circ t_3(\alpha)$ for every $\alpha \in For^{\circ}$. Observe that, if β is an axiom of **eCPL** different from (exp) (see Remark 24) then $t_3(\beta)$ is an axiom of **Cia**. On the other hand, if $\beta = \delta \rightarrow (\neg\delta \rightarrow \gamma)$ is an instance of (exp) then $t_3(\beta) = t_3(\delta) \rightarrow (\neg t_3(\delta) \rightarrow t_3(\gamma))$, and the latter is provable in **Cia** from (bc1) and $\circ t_3(\delta)$. Thus $t_3(\beta)$ is a theorem of **Cia**. Note also that any application of *modus ponens* in **eCPL** is transformed into an application of *modus ponens* in **Cia**. Consequently, given a derivation $\alpha_1, \ldots, \alpha_n = \alpha$ of α in **eCPL**, the finite sequence of formulas $t_3(\alpha_1), \ldots, t_3(\alpha_n) = t_3(\alpha)$ can be transformed into a derivation of $t_3(\alpha)$ in **Cia**. This shows that $\vdash_{\mathbf{eCPL}} \alpha$ implies $\vdash_{\mathbf{Cia}} t_3(\alpha)$.

In order to prove the converse, consider the definition of an adequate valuation semantics for **Cia**, adding to the clauses of a valuation semantics for **Ci** (see Definition 82) the clause (vC7) of Example 58. Now, given an **eCPL**-valuation v, consider the mapping $v' : \ \mathcal{P} \cup \{\neg p \ : \ p \in \mathcal{P}\} \longrightarrow \mathbf{2}$

such that $v'(p) = 1$ for every $p \in \mathcal{P}$, and $v'(\neg p) = 1$ iff $v(p) = 0$. Define now $v'(\circ p) = 1$ iff $v'(\neg p) = 0$, and extend v' homomorphically to the remaining formulas in For° using the matrices for **eCPL**. That is, for formulas other than p, $\neg p$ and $\circ p$ (for $p \in \mathcal{P}$) the mapping v' is defined as a classical valuation and moreover satisfies $v'(\circ\alpha) = 1$ for every non-atomic α. It is easy to see that this v' is indeed a **Cia**-valuation. An induction on the complexity $l(\alpha)$ of α shows that $v(\alpha) = v'(t_3(\alpha))$ for every $\alpha \in For^{\circ}$. Finally, suppose that $\nvdash_{\mathbf{eCPL}} \alpha$. Then, there is some **eCPL**-valuation v such that $v(\alpha) = 0$. But then, by the argument above, there is some **Cia**-valuation v' such that $v'(t_3(\alpha)) = 0$ and so $\nvdash_{\mathbf{Cia}} t_3(\alpha)$. ■

Straightforward adaptations of the above argument show that the same t_3 acts as a conservative translation between **eCPL** and all logics defined in item (ii) of Definition 108. So, in order to perform 'classical inferences' within such logics (and even within C_1, as it coincides with **Cila**), it suffices to translate every atomic formula p into $\circ p$.

Axioms (ca1)–(ca3) of Definition 108 describe a certain form of propagation of consistency through conjunction. There are several other sensible ways of allowing consistency or inconsistency to propagate. It also makes sense to think, for instance, of propagation of consistency through disjunction:

DEFINITION 112.
(i) The logic **Cio** is obtained by the addition to **Ci** of the axiom schemas

(co1) $(\circ\alpha \vee \circ\beta) \to \circ(\alpha \wedge \beta)$;

(co2) $(\circ\alpha \vee \circ\beta) \to \circ(\alpha \vee \beta)$;

(co3) $(\circ\alpha \vee \circ\beta) \to \circ(\alpha \to \beta)$.

(ii) The logic **Cilo** is obtained by the addition to **Cio** of the axiom schema (cl) (see Definition 102 and Theorem 104). This logic was introduced in [Béziau, 1990] and was studied under the name C_1^+ in [da Costa *et al.*, 1995]. As in Definition 108, several other logics can be defined from **Cio** by tinkering with axioms (cf), (cb) and (ce). ■

Obviously, C_1^+ is a deductive extension of C_1. The weaker requirement to obtain consistency of a complex formula, namely, the consistency of at least one of its components, reflects in the following immediate stronger result:

THEOREM 113.
If $\Gamma \vdash_{\mathbf{Cio}} \circ\beta$ for some subformula β of α, then $\Gamma \vdash_{\mathbf{Cio}} \circ\alpha$. ■

Similar arguments to those given in the proof of Theorem 111 will show again that the same t_3 is also a conservative translation between **eCPL** and the logics presented in Definition 112.

On what concerns the interdefinability of the binary connectives with the help of our primitive paraconsistent negation (compare with Theorem 57), we now can prove the following extra rules:

THEOREM 114. In **Cia** the following holds:
(ix) $\neg(\neg\alpha \wedge \neg\beta) \vdash_{\mathbf{mbC}} (\alpha \vee \beta)$.
In **Cio** the following hold:
(vi) $\neg(\alpha \wedge \neg\beta) \vdash_{\mathbf{Cio}} (\alpha \to \beta)$;
(vii) $\neg(\alpha \to \beta) \vdash_{\mathbf{Cio}} (\alpha \wedge \neg\beta)$;
(xi) $\neg(\neg\alpha \vee \neg\beta) \vdash_{\mathbf{Cio}} (\alpha \wedge \beta)$. ■

From Theorem 114(vii) and Theorem 44(ii) we can conclude that the replacement property (RP) (recall Remark 43) does not hold for any extension of **Cio**. However, a restricted form of this property can be attained:

REMARK 115. Say that a logic **L** *allows for replacement with respect to* $\approx$ when $\approx$ is a binary connective such that, for every formula $\varphi(p_0, \ldots, p_n)$ and formulas $\alpha_0, \ldots, \alpha_n$, $\beta_0, \ldots, \beta_n$:

(RRP) $(\Vdash_{\mathbf{L}} \alpha_0 \approx \beta_0)$ and $\ldots$ and $(\Vdash_{\mathbf{L}} \alpha_n \approx \beta_n)$ implies
$$\Vdash_{\mathbf{L}} \varphi(\alpha_0, \ldots, \alpha_n) \approx \varphi(\beta_0, \ldots, \beta_n).$$

Any such connective, when it exists, will be called a *congruence* of **L**. Notice that, for our present logics, full replacement holds exactly when $\leftrightarrow$ is a congruence. ■

THEOREM 116. A congruence in **Cio** can be defined by setting $\alpha \approx \beta \stackrel{\text{def}}{=} (\alpha \leftrightarrow \beta) \wedge (\circ\alpha \wedge \circ\beta)$.

Proof. A semantic proof for **Cilo** was done in theorem 3.21 of [da Costa *et al.*, 1995]. A similar argument, adapted for **Cio**, can be found in Fact 3.81 of [Carnielli and Marcos, 2002]. ■

Again, on what concerns the semantic interpretation of the above logics:

THEOREM 117. The logics **Cia**, **Cila**, **Cio** and **Cilo** are not characterizable by finite matrices.

Proof. The result can be proven for **Cia** and **Cila** using either Theorem 46 or 47. For **Cio** and **Cilo** only this last theorem will do the job. ■

REMARK 118. The logic **Cibae** (Definitions 108), an obvious extension of C_1, received an adequate interpretation in terms of possible-translations semantics in [Carnielli, 2000] and in [Marcos, 1999]. In the latter study, all the other logics from Definitions 108 and 112 have also received adequate possible-translations semantics. ■

We end this subsection with an axiomatization of two important **LFI**s through the regulation of their ability to propagate inconsistency.

THEOREM 119. The logic **LFI1** described in Example 14 is axiomatized by adding to **Cie** (check Definition 97) the following axiom schemas:

(cj1) $\bullet(\alpha \wedge \beta) \leftrightarrow ((\bullet\alpha \wedge \beta) \vee (\bullet\beta \wedge \alpha))$;

(cj2) $\bullet(\alpha \vee \beta) \leftrightarrow ((\bullet\alpha \wedge \neg\beta) \vee (\bullet\beta \wedge \neg\alpha))$;

(cj3) $\bullet(\alpha \rightarrow \beta) \leftrightarrow (\alpha \wedge \bullet\beta)$.

The logic $\mathbf{P}^1$ described in Example 15 is axiomatized by adding to **Ci** (check Definition 97) the following axiom schemas:

(cz) $\circ\alpha$ (for α non-atomic) ■

In the last theorem, note that (cz), in fact, consists of five axiom schemas, one for each connective in the signature Σ°. The logic $\mathbf{P}^1$ describes an extreme case of propagation of consistency into complex formulas, where no premises are needed so as to guarantee it.

*5.3 **LFI**s that are maximal relative to **CPL***

[da Costa, 1974] suggests the 'natural' features that a paraconsistent logic should enjoy. One of these is that a paraconsistent logic should contain the most part of the schemas and rules of the classical propositional logic which do not interfere with paraconsistency. Following [Marcos, 2001], one way of implementing this feature would be by requiring paraconsistent logics to be maximal subsystems of classical logic.

The following notion of maximality among logics can be used to analyze how close we are to having 'most of classical logic' inside paraconsistent systems:

DEFINITION 120. Let **L**1 and **L**2 be two logics written in the same signature. Then, **L**2 is said to be *maximal relative to* **L**1 if:
(i) **L**1 is an extension of **L**2;
(ii) if $\vdash_{\mathbf{L1}} \alpha$ but $\nvdash_{\mathbf{L2}} \alpha$, then the logic obtained from **L**2 by adding α as a new axiom schema coincides with **L**1.
When **L**1 is clear from the context, we simply say that a logic **L**2 satisfying conditions (i) and (ii) is *maximal*. ■

The above introduced concept has many instances. It is well-known, for instance, that each Łukasiewicz's logic Ł_m, for $m > 2$, is maximal relative to **CPL**, the classical propositional logic, if and only if $(m-1)$ is a prime number. Also, **CPL** is maximal relative to the trivial logic, a logic in which all formulas are provable. On the other hand it is also well-known that intuitionistic logic is not a maximal fragment of **CPL**, and there exists indeed an infinite number of intermediate logics between them. On what concerns the **C**-systems presented this far, only the logic **LFI1** and the logic $\mathbf{P}^1$, described in Examples 14 and 15, and Theorem 119, are maximal relative to **CPL**, or relative to **eCPL**, the extended version of **CPL** introduced in the beginning of Subsection 3.6. In particular, the logic C_1 (or, equivalently,

Cila — recall Remark 109), despite being the strongest logic introduced by da Costa on his first hierarchy of paraconsistent logics, is properly extended by $\mathbf{P}^1$ and fails to be maximal. Therefore, none of the logics C_n presented in [da Costa, 1974] respects the requirement of containing the most of classical logic demanded in that very paper. The same observation in fact is true also about the stronger logic called C_1^+, introduced later on (Definition 112).

Now we explore the intuitions underlying the three-valued maximal **C**-systems $\mathbf{P}^1$ and **LFI1** showing how to generate a whole class of three-valued maximal paraconsistent logics. Looking for models for contradictory and non-trivial theories, we start with non-trivial interpretations under which both some formula α and its negation $\neg\alpha$ would be simultaneously satisfied. A natural choice lies in the many-valued domain, namely in logics given by finite-valued matrices. Since we want to preserve classical theses as much as possible, the values of the connectives with classical (0 and 1) inputs will have classical outputs. Suppose we just introduce a third intermediate value $\frac{1}{2}$, besides true (1) and false (0), where $D = \{1, \frac{1}{2}\}$ is the set of designated values. Then there are two possible classic-like truth-tables for a negation validating α and $\neg\alpha$ simultaneously, for some α, namely:

	$\neg$
1	0
$^1/_2$	$^1/_2$ or 1
0	1

With respect to the other connectives of the signature Σ (since we try to keep them as classical as possible), we add the following higher-level classic-like requirements:

(C$\wedge$) $(x \wedge y) \in D$ iff $x \in D$ and $y \in D$;

(C$\vee$) $(x \vee y) \in D$ iff $x \in D$ or $y \in D$;

(C$\to$) $(x \to y) \in D$ iff $x \notin D$ or $y \in D$.

The above constrains leave us with the following options:

$\wedge$	1	$^1/_2$	0
1	1	$^1/_2$ or 1	0
$^1/_2$	$^1/_2$ or 1	$^1/_2$ or 1	0
0	0	0	0

$\vee$	1	$^1/_2$	0
1	1	$^1/_2$ or 1	1
$^1/_2$	$^1/_2$ or 1	$^1/_2$ or 1	$^1/_2$ or 1
0	1	$^1/_2$ or 1	0

$\to$	1	$^1/_2$	0
1	1	$^1/_2$ or 1	0
$^1/_2$	$^1/_2$ or 1	$^1/_2$ or 1	0
0	1	$^1/_2$ or 1	1

This yields 2^3 options for conjunctions, 2^5 options for disjunctions, 2^4 options for implications, and, as stated above, 2^1 options for negations, adding up to 2^{13} $(= 8,192)$ possible logics to deal with, in the signature Σ. Of course, not all those logics are necessarily 'interesting'. We can upgrade each of those logics into an **LFI** by considering the signature $\Sigma^{\circ\bullet}$ and adding the following tables for consistency and inconsistency operators:

	$\circ$	$\bullet$
1	1	0
$^1/_2$	0	1
0	1	0

This means that the consistent models are the ones given by classical valuations, and only those.

DEFINITION 121. The collection of logics defined by the above matrices, with designated values $D = \{1, \frac{1}{2}\}$, will be called $8Kb$.

Clearly, every logic in $8Kb$ is a fragment of **eCPL**, the extended classical propositional logic, if we consider in **eCPL** the usual definition of the inconsistency connective as the negation of the consistency connective. Note also that the logic *Pac* (see Example 13) does not belong to $8Kb$, because it cannot define the connectives $\circ$ and $\bullet$. On the other hand, its conservative extension **LFI1** can, and the latter belongs indeed to $8Kb$. The three-valued logic $\mathbf{P}^1$ is also in $8Kb$, and we already know that these two logics are axiomatizable by adding suitable axioms to the axiomatization of **Ci** (see Theorem 119). As shown in [Marcos, 2000], this same method can be extended to the whole $8Kb$:

THEOREM 122. (i) Every logic in $8Kb$ is an axiomatic extension of **Cia**.
(ii) All the logics in $8Kb$ are distinct from each other, and they are all maximal relative to **eCPL**.
(iii) All the logics in $8Kb$, and their fragments, are boldly paraconsistent.■

It is just a combinatorial divertissement to check the following:

THEOREM 123. All the $8,192$ logics in $8Kb$ are **C**-systems extending **Cia** (cf. Definition 108). Out of these, $7,680$ are in fact **dC**-systems, being able to define $\circ$ and $\bullet$ in terms of the other connectives (therefore, maximal relative to **CPL**, and not only to **eCPL**). Of these, $4,096$ are able to define $\circ\alpha$ as $\neg(\alpha \wedge \neg\alpha)$, and so all of these do extend C_1 (that is, **Cila**). Of the $7,680$ logics which are **dC**-systems, $1,680$ extend **Cio** (cf. Definition 112), and 980 of the latter are able to define $\circ\alpha$ as $\neg(\alpha \wedge \neg\alpha)$, so that these 980 logics do extend C_1^+ (that is, **Cilo**). ■

The replacement property (RP) had already been shown to fail for our foremost logic samples from the $8Kb$. Indeed, the proof of items (iv) and

(v) of Theorem 42 showed that both **LFI1** and $\mathbf{P}^1$ fail (RP). This negative feature can be generalized:

THEOREM 124. (RP) cannot hold in any of the logics in $8Kb$.

Proof. This is true in general for in any extension of **Cia**, as we may conclude from Theorem 78(ii) and Theorem 114(ix). To complete the proof, recall Theorem 122(i).

You will also be able to check the above result, alternatively, using the classical negation from Theorem 125 and item (i) of Theorem 44. ■

As a consequence of Theorem 124 the logics in $8Kb$ are not suitable to an algebraization by means of a direct Lindenbaum-Tarski-style procedure. However, the next results guarantee that all of them are algebraizable in the sense of Blok-Pigozzi (cf. [Blok and Pigozzi, 1989]).

THEOREM 125. Each one of the logics in $8Kb$ defines the following matrix for classical negation and at least one of the congruences below:

	$\sim$
1	0
$^1/_2$	0
0	1

$\equiv$	1	$^1/_2$	0
1	1	0	0
$^1/_2$	0	$^1/_2$ or 1	0
0	0	0	1

Proof. It is possible to define $\bot$ either as $(\alpha \wedge (\neg\alpha \wedge \circ\alpha))$ or as $(\circ\alpha \wedge \neg\circ\alpha)$, for any formula α. Then, we can define $\sim\alpha$ either as $(\neg\alpha \wedge \circ\alpha)$ or as $(\alpha \to \bot)$. One of the above congruences $(\alpha \equiv \beta)$ can always be defined by $((\alpha \leftrightarrow \beta) \wedge (\circ\alpha \leftrightarrow \circ\beta))$. In cases we choose to have $(\frac{1}{2} \equiv \frac{1}{2}) = 1$, we can define the new congruence by setting $(\alpha \bowtie \beta) \stackrel{\text{def}}{=} \sim\sim(\alpha \equiv \beta)$. ■

The following theorem generalize a result obtained in [Lewin *et al.*, 1990] for the logic $\mathbf{P}^1$:

THEOREM 126. All the logics in $8Kb$ are Blok-Pigozzi algebraizable.

Proof. Consider $\Delta(p_0, p_1) = \{(p_0 \equiv p_1)\}$ or $\Delta = \{(p_0 \bowtie p_1)\}$, where $\equiv$ and $\bowtie$ are defined as in the proof of the Theorem 125. Take the sets

$$\delta(p_0) = \{((p_0 \to p_0) \to p_0)\}, \quad \varepsilon(p_0) = \{(p_0 \to p_0)\}$$

and check that the conditions of [Blok and Pigozzi, 1989] are satisfied. ■

On what concerns the expressibility of the class $8Kb$ and the distinguished logics $\mathbf{P}^1$ and **LFI1**, we can check the following results:

THEOREM 127.
(i) The matrices of $\mathbf{P}^1$ can be defined inside of any of the logics in $8Kb$.
(ii) All the matrices in $8Kb$ can be defined inside of **LFI1**.

Proof. (i) Fix some logic **L** belonging to $8Kb$. Let $\wedge, \vee, \rightarrow, \neg, \circ$ and $\bullet$ be its connectives, and let $\sim$ be the classical negation defined inside **L** as in Theorem 125. Then, the $\mathbf{P}^1$-negation of a formula α can be defined in **L** as $\sim\sim\neg\alpha$. The $\mathbf{P}^1$-conjunction of some given formulas α and β can be defined in **L** either as $\sim\sim(\alpha \wedge \beta)$ or as $(\sim\sim\alpha \wedge \sim\sim\beta)$. A definition in the same vein applies to both disjunction and implication. Note that the matrices in **L** for the connectives $\circ$ and $\bullet$ already coincide with those of $\mathbf{P}^1$.
(ii) One proof of this property can be found in [Avron, 1999]. A constructive proof can be found in [Marcos, 1999] and [Carnielli *et al.*, 2000]. ■

COROLLARY 128. (i) The logic $\mathbf{P}^1$ can be conservatively translated into any of the logics in $8Kb$. (ii) Any of the logics in $8Kb$ can be conservatively translated into **LFI1**. ■

As argued in [Avron, 1991], the logic **LFI1** has several properties that justify its role as one of the most 'natural' 3-valued paraconsistent logics. Theorem 127(ii) and Corollary 128(ii) show already how linguistically and deductively expressive this logic is.

A last note on algebraization. We had several occasions above to see how replacement fails for many of our **LFI**s. This often makes it difficult to provide algebraic counterparts in the usual sense for those logics. However, it is interesting to observe that some kind of algebraic treatment for some wilder **C**-systems has been proposed and studied, for instance, in [Carnielli and de Alcantara, 1984] and [Seoane and de Alcantara, 1991] (for a partial survey, check the section 3.12 of [Carnielli and Marcos, 2002]). Additionally, an approach based on an idea similar to that of a possible-translations structure for algebraizing **LFI**s was presented in [Bueno *et al.*, 2004].

6 CONCLUSIONS AND FURTHER PERSPECTIVES

In the last part of this chapter we recall some definitions and results obtained and described above, and point to some interesting new problems and research directions connected to what has been presented.

From Section 3 on, some of the possibilities for the formalization and understanding of the relationship between the concepts of consistency, inconsistency, contradictoriness and triviality were explored at a very general level. Assuming that consistency could be expressed inside some paraconsistent logics, and assuming furthermore that the consistency of a given formula would legitimate its explosive character (that is, assuming a Gentle Principle of Explosion, see (9)), we have given in Subsection 3.1 (Definition 18) a general definition of a Logic of Formal Inconsistency, **LFI**. To actualize that definition (in a finitary way), we have started our study from the logic **mbC**, a very weak **C**-system based on classical logic (recall Definition 34), constructing all the remaining **C**-systems as extensions of **mbC**.

Some specific extensions of **mbC** singled out a subclass of the **C**-systems in which the connectives '$\circ$' for consistency and '$\bullet$' for inconsistency are definable by means of other connectives. The members of this class were called **dC**-systems (recall Definition 25).

We briefly recall some consequences of our approach to formal (in)consistency: There are consistent and inconsistent logics. The inconsistent ones may be either paraconsistent or trivial, but not both. Let us say that a theory *has non-trivial models* only if these models do not assign designated values to all formulas. Thus, the theories of a consistent logic have non-trivial models if and only if they are non-contradictory. Paraconsistent logics will have non-trivial models for some of its contradictory theories. For each formula α of a logic **L**, the consistency $\circ\alpha$ of α constitutes what should be added to an α-contradictory theory in order to make it explosive, and consequently trivial. If the answer is 'there is nothing to be added', then α is already consistent in **L**. This means that, as expected, a logic is consistent if all its formulas are consistent.

It will be clear now to the reader that there are many more examples of **C**-systems besides the logics C_n of da Costa and other logics axiomatized in a more or less similar fashion. The general idea is to express consistency and inconsistency inside a logic, at its object-language level. This approach allows us to collect in a single class of **LFI**s logics as diverse as the C_n and $\mathbf{P}^1$, $\mathbf{J}_3$ (renamed **LFI1**), and Jaśkowski's 'discussive' paraconsistent logic **D2** (cf. Example 19). Even normal modal logics in a convenient signature can be very naturally regarded as **dC**-systems. This bears on the relationship between negations and modalities, which reflects upon the possibilities of defining paraconsistent negations in modal environments, as studied by [Vakarelov, 1989], [Došen, 1986], [Béziau, 2002b] and [Marcos, 2004a].

The fact that so many logics with diverse motivations and technical features can be recast as a **dC**-systems opens an interesting question: To check whether other logics in the literature on paraconsistent logics could be characterized as **C**-systems, or, in general, as **LFI**s. Another related question is the following: How to enrich a given paraconsistent logic in order to turn it into an **LFI**? This was done by the logic **LFI1** (also known as **CLuNs**, or $\mathbf{J}_3$) with respect to the logic *Pac* (see Example 14). Consider now the three-valued *closed set logic* studied in [Mortensen, 1995]. This logic consists of **LFI1**'s matrices of conjunction and of disjunction, plus the matrix of negation of $\mathbf{P}^1$, where 0 is the only non-designated value. The addition of appropriate matrices of implication and of a consistency operator would enrich the closed-set logic, and the resulting logic would belong to the collection $8Kb$ of three-valued maximal paraconsistent logics (recall Definition 121). Is it possible to define such a consistency operator in the closed-set logic from the start, or must it be introduced by an extension and a non-eliminable definition? In either case, what would be its topological or set-theoretical significance?

The question of the duality between intuitionistic and paraconsistent logics, not explored in this chapter, is also worth mentioning. The concept of dual-intuitionism was already mentioned in the 40s by K. Popper, cf. [Popper, 1948], more or less at the same time as paraconsistency was being engendered. More recently, dual-intuitionism and dual-paraconsistency have been studied, for example, in [Sylvan, 1990], [Urbas, 1996] and [Brunner and Carnielli, 2004]. The logics that are dual to paraconsistent are sometimes called 'paracomplete' (cf. [Loparić and da Costa, 1984]). Exploring the issue of duality, a natural question that appears concerns the notions that are dual to consistency and inconsistency, notions that one might dub 'determinedness' and 'undeterminedness'. Some initial explorations in that direction, and the related *Logics of Formal Undeterminedness*, can be found in [Marcos, 2004a].

Apparently, in the 40s, defenders of dual-intuitionism and paraconsistency independently realized that there should be a logic for general reasoning from hypotheses, accepting in certain cases some propositions and their negations as true (in the case of paraconsistency), or retaining some propositions and their negations as not falsified (in the case of falsificationism). Indeed, there seems to be some common grounds connecting paraconsistency and the falsificationist program in Philosophy of Science, and that line of research seems worth pursuing. Similarly, the dual paracomplete logics could have a contribution to make to the study of verificationism in science.

Applications of **LFI**s to yet other fields of philosophy seem promising and start to appear. In [Costa-Leite, 2003] and [Carnielli *et al.*, 2004] the possibility of employing the connectives of consistency and inconsistency on the understanding and solution of epistemological problems related to the paradox of knowability is investigated. In [Marcos, 2004e] the use of a consistency-like modal connective for the modelling of the metaphysical notion of essence is tackled, and in that environment inconsistency turns out to be a mere sort of accident.

Another important issue concerns the incompleteness results in Arithmetic. Recall that Gödel's incompleteness theorems are based on the identification of 'consistency' and 'non-contradictoriness'. What would be the consequences if we started from the general notion of consistency hereby proposed (recall Definition 4)? Would it still be possible to reproduce Gödel's arguments? Quite possibly, his arguments would be rescued at the cost of assuming consistency (in our sense) of several formulas representing assumptions that would then become more explicit, and consequently open to debate. In the same spirit, it should be interesting to analyze the combination of **LFI**s with modal logics of provability. In [Boolos, 1996], consistency is intended as a kind of opposite of provability. Using this idea, if the negation of a formula cannot be proved, then it is consistent with what was proved; a still weaker notion, connected to 'logical independence', would be to consider a formula to be consistent when neither this formula nor its negation can be proved. The insinuated exchange between Logics of Formal Inconsistency and Logics of Provability, in fact, seems interesting and deserves further research.

As it was noted in the literature, it seems that most interesting problems related to paraconsistency appear already at the propositional level. It is possible though to extend a given propositional paraconsistent logic to higher levels using combination techniques such as fibring, if only we choose the right abstraction level to express our logics. See, for instance, [Caleiro and Marcos, 2001], where the logic C_1 is given a first-order version which coincides with the original one from [da Costa, 1993]. An interesting possibility about first-order versions of paraconsistent logics in general, and especially of first-order **LFI**s, is the investigation of consistent yet ω-inconsistent theories (also related to Gödel's theorems).

Some other items for future research, already hinted at along the text, are the following. From Theorem 76 we know that, in extensions of **mCi**, the formulas causing controllable explosion (Definition 8(ii)) coincide with the provably consistent formulas, that is, theorems of the form $\circ\alpha$. On the other hand, **mbC** does not have provably consistent formulas (see Theorem 39). So, is the logic **mbC** (see Definition 34) controllably explosive? On another trail, we have seen that there are extensions of **mbC** for which the replacement property holds good (see Remark 45), and we have seen that to find extensions of **mCi** with that same property all one needs to do is to find logics that respect a certain rule (EC) (see Subsection 3.2 and Theorem 79). Can we circumvent negative results such as Theorems 44 and 78 and find interesting extensions of **mCi** satisfying the replacement property? At any rate, the study of extensions of **mbC** that do not extend **mCi** seems a very attractive enterprise. On yet another direction, what other uses could we give to our semantical tools (valuations and possible-translations semantics)? The results about uncharacterizability by finite matrices in Theorems 46 and 47 seem very interesting and quite complementary to each other, but they cannot help us in establishing that logics such as **Cioe** do not have adequate finite-valued matrices. Can we find other flexible and wide-ranging results to the same purpose?

Finally, we have started our work in this chapter from a traditional syntactical perspective. But we have soon shown that alternative semantical and syntactical approaches were possible. In particular, we have given a few illustrations of a general method that permits us to deal with **C**-systems in terms of tableaux. The first general method to such an effect was proposed in [Carnielli and Marcos, 2001b]. A more general method to obtain tableau procedures for logics endowed with a certain type of two-valued (even non-truth-functional) semantics was introduced in [Caleiro *et al.*, 2004]. These techniques have been used here in Subsections 3.5 and 4.2 to obtain new adequate tableau systems for the logic C_1, as well as for **mbC** and **mCi**. The possibility of further exploring and refining this kind of approach seems promising for applications of **LFI**s in database theory (see Example 14), an area of research critically sensible to contradictions.

7 LIST OF AXIOMS AND SYSTEMS

We list here all the principles, axioms and systems studied throughout the chapter, indicating the place where they were introduced in the text.

Principles

(1) Principle of Non-Contradiction : Subsection 2.1
(2) Principle of Non-Triviality : Subsection 2.1
(3) Principle of Explosion, or *Pseudo-Scotus*, or *ex contradictione sequitur quodlibet* : Subsection 2.1
(4) Paraconsistent logic (first definition) : Subsection 2.2
(5) Paraconsistent logic (second definition) : Subsection 2.2
(6) Paraconsistent logic (third definition) : Subsection 2.2
(7) Principle of *Ex Falso Sequitur Quodlibet* : Subsection 2.2
(8) Supplementing Principle of Explosion : Subsection 2.2
(9) Gentle Principle of Explosion : Subsection 3.1
(10) Finite Gentle Principle of Explosion : Subsection 3.1

Axioms, Rules and Metaproperties

(Ax1)–(Ax11) : Definition 23
(bc1) : Definition 23, Definition 34
(bc1)' : Theorem 94
(bc2) : Theorem 94
(ca1)–(ca3) : Definition 23, Definition 108
(cb) : Subsection 5.1
$(\text{cc})_n$: Definition 72
$(\text{cc})'_n$: Theorem 94
(cd) : Subsection 5.1
(ce) : Subsection 4.4
(cf) : Subsection 4.4
(ci) : Definition 72
(ci)' : Theorem 94
(cj1)–(cj3) : Theorem 119
(cl) : Definition 102

Systems

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BIBLIOGRAPHY

[Arruda, 1975] A. I. Arruda. Remarques sur les systèmes C_n. *Comptes Rendus de l'Académie de Sciences de Paris (A-B)*, 280:1253–1256, 1975.

[Arruda, 1980] A. I. Arruda. A survey of paraconsistent logic. In A. I. Arruda, R. Chuaqui, and N. C. A. da Costa, editors, *Mathematical Logic in Latin America: Proceedings of the IV Latin American Symposium on Mathematical Logic*, pages 1–41. North-Holland, 1980.

[Asenjo, 1966] F. G. Asenjo. A calculus of antinomies. *Notre Dame Journal of Formal Logic*, 7:103–105, 1966.

[Avron, 1986] A. Avron. On an implication connective of *RM*. *Notre Dame Journal of Formal Logic*, 27:201–209, 1986.

[Avron, 1991] A. Avron. Natural 3-valued logics — Characterization and proof theory. *The Journal of Symbolic Logic*, 56(1):276–294, 1991.

[Avron, 1999] A. Avron. On the expressive power of three-valued and four-valued languages. *Journal of Logic and Computation*, 9:977–994, 1999.

[Avron, 2002] A. Avron. On negation, completeness and consistency. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, 2^{nd} Edition*, volume 9, pages 287–319. Kluwer Academic Publishers, 2002.

[Batens and De Clercq, 2000] D. Batens and K. De Clercq. A rich paraconsistent extension of full positive logic. Draft, 2000.

[Batens, 1980] D. Batens. Paraconsistent extensional propositional logics. *Logique et Analyse*, 90/91:195–234, 1980.

[Béziau, 1990] J.-Y. Béziau. Logiques construites suivant les méthodes de da Costa. I. Logiques paraconsistantes, paracompletes, non-alèthiques construites suivant la première méthode de da Costa. *Logique et Analyse (N.S.)*, 131/132:259–272, 1990.

[Béziau, 1994] J.-Y. Béziau. Théorie legislative de la négation pure. *Logique et Analyse*, 147/148:209–225, 1994.

[Béziau, 1999] J.-Y. Béziau. La véritable portée du théorème de Lindenbaum-Asser. *Logique et Analyse*, 167/168:341–359, 1999.

[Béziau, 2002a] J.-Y. Béziau. Are paraconsistent negations negations? In W. A. Carnielli, M. E. Coniglio, and I. M. L. D'Ottaviano, editors, *Paraconsistency - the Logical Way to the Inconsistent*, volume 228 of *Lecture Notes in Pure and Applied Mathematics*, pages 465–486, New York, 2002. Marcel Dekker.

[Béziau, 2002b] J.-Y. Béziau. *S*5 is a paraconsistent logic and so is first-order classical logic. *Logical Studies*, 9:301–309, 2002.

[Blok and Pigozzi, 1989] W. J. Blok and D. Pigozzi. Algebraizable Logics. *Memoirs of the American Mathematical Society*, 396, 1989.

[Bobenrieth-Miserda, 1996] A. Bobenrieth-Miserda. *Inconsistencias ¿Por qué no? Un estudio filosófico sobre la lógica paraconsistente*. Tercer Mundo Editores, Santafé de Bogotá, 1996.

[Boolos, 1996] G. Boolos. *The Logic of Provability*. Cambridge University Press, 1996.

[Brunner and Carnielli, 2004] A. B. M. Brunner and W. A. Carnielli. Anti-intuitionism and paraconsistency. *Journal of Applied Logics*, 2004. In print. Preprint available at *CLE e-Prints*, 3(1), 2003.
URL = `http://www.cle.unicamp.br/e-prints/vol_3,n_1,2003.htm`.

[Bueno *et al.*, 2004] J. Bueno, M. E. Coniglio, and W. A. Carnielli. Finite algebraizability via possible-translations semantics. In W. A. Carnielli, F. M. Dionísio, and P. Mateus, editors, *Proceedings of the Workshop on Combination of Logics: Theory and applications* (CombLog'04), held in Lisbon, PT, 28–30 July 2004, pages 79–86. Departamento de Matemática, Instituto Superior Técnico, 2004. Preprint available at URL = `http://www.cs.math.ist.utl.pt/comblog04/abstracts/bueno.pdf`.

[Bueno, 1999] O. Bueno. Truth, quasi-truth and paraconsistency. In W. A. Carnielli and I. M. L. D'Ottaviano, editors, *Advances in Contemporary Logic and Computer Science*, volume 235 of *Contemporary Mathematics Series*, pages 275–293. American Mathematical Society, 1999.

[Caleiro and Marcos, 2001] C. Caleiro and J. Marcos. Non-truth-functional fibred semantics. In H. R. Arabnia, editor, *Proceedings of the 2001 International Conference on Artificial Intelligence (IC-AI'2001)*, volume II, pages 841–847. CSREA Press, 2001.

[Caleiro *et al.*, 2004] C. Caleiro, W. A. Carnielli, M. E. Coniglio, and J. Marcos. How many logical values are there? Dyadic semantics for many-valued logics. Draft, 2004. Forthcoming.

[Carnielli and de Alcantara, 1984] W. A. Carnielli and L. P. de Alcantara. Paraconsistent algebras. *Studia Logica*, 43(1/2):79–88, 1984.

[Carnielli and Lima-Marques, 1992] W. A. Carnielli and M. Lima-Marques. Reasoning under inconsistent knowledge. *Journal of Applied Non-Classical Logics*, 2(1):49–79, 1992.

[Carnielli and Marcos, 1999] W. A. Carnielli and J. Marcos. Limits for paraconsistent calculi. *Notre Dame Journal of Formal Logic*, 40(3):375–390, 1999.

[Carnielli and Marcos, 2001a] W. A. Carnielli and J. Marcos. *Ex contradictione non-sequitur quodlibet*. In *Proceedings of the Advanced Reasoning Forum Conference*, volume I of *Bulletin of Advanced Reasoning and Knowledge*, pages 89–109, 2001.

[Carnielli and Marcos, 2001b] W. A. Carnielli and J. Marcos. Tableau systems for logics of formal inconsistency. In H. R. Arabnia, editor, *Proceedings of the 2001 International Conference on Artificial Intelligence (IC-AI'2001)*, volume II, pages 848–852. CSREA Press, 2001.

[Carnielli and Marcos, 2002] W. A. Carnielli and J. Marcos. A taxonomy of **C**-systems. In W. A. Carnielli, M. E. Coniglio, and I. M. L. D'Ottaviano, editors, *Paraconsistency - the Logical Way to the Inconsistent*, volume 228 of *Lecture Notes in Pure and Applied Mathematics*, pages 1–94, New York, 2002. Marcel Dekker. Preprint available at *CLE e-Prints*, 1(5), 2001.
URL = `http://www.cle.unicamp.br/e-prints/abstract_5.htm`.

[Carnielli *et al.*, 2000] W. A. Carnielli, J. Marcos, and S. de Amo. Formal inconsistency and evolutionary databases. *Logic and Logical Philosophy*, 8:115–152, 2000. Preprint available at *CLE e-Prints*, 1(6), 2001.
URL = `http://www.cle.unicamp.br/e-prints/abstract_6.htm`.

[Carnielli *et al.*, 2004] W. A. Carnielli, M. E. Coniglio, and A. Costa-Leite. Solving the knowability paradox using modal LFIs. Forthcoming, 2004.

[Carnielli, 1990] W. A. Carnielli. Many-valued logics and plausible reasoning. In *Proceedings of the Twentieth International Symposium on Multiple-Valued Logic*, pages 328–335, Charlotte, NC, USA, 1990. IEEE Computer Society.

[Carnielli, 2000] W. A. Carnielli. Possible-translations semantics for paraconsistent logics. In D. Batens, C. Mortensen, G. Priest, and J. P. Van Bendegem, editors, *Frontiers of Paraconsistent Logic: Proceedings of the I World Congress on Paraconsistency*, Logic and Computation Series, pages 149–163. Baldock: Research Studies Press, King's College Publications, 2000.

[Coniglio and Carnielli, 2002] M. E. Coniglio and W. A. Carnielli. Transfers between logics and their applications. *Studia Logica*, 72(3):367–400, 2002. Preprint available at *CLE e-Prints*, 1(4), 2001.
URL = `http://www.cle.unicamp.br/e-prints/abstract_4.htm`.

[Costa-Leite, 2003] A. Costa-Leite. Paraconsistência, modalidades e cognoscibilidade (Paraconsistency, modalities and cognoscibility, in Portuguese). Master's thesis, State University of Campinas (Unicamp), Campinas, 2003.

[da Costa and Alves, 1977] N. C. A. da Costa and E. Alves. A semantical analysis of the calculi C_n. *Notre Dame Journal of Formal Logic*, 18(4):621–630, 1977.

[da Costa and Béziau, 1993] N. C. A. da Costa and J.-Y. Béziau. Carnot's logic. *Bulletin of the Section of Logic*, 22(3):98–105, 1993.

[da Costa and Marconi, 1989] N. C. A. da Costa and D. Marconi. An overview of paraconsistent logic in the 80s. *The Journal of Non-Classical Logics*, 6(1):5–32, 1989.

[da Costa *et al.*, 1995] N. C. A. da Costa, J.-Y. Béziau, and O. A. S. Bueno. Aspects of paraconsistent logic. *Bulletin of the IGPL*, 3(4):597–614, 1995.

[da Costa, 1959] N. C. A. da Costa. Observações sobre o conceito de existência em matemática. *Anuário da Sociedade Paranaense de Matemática*, 2:16–19, 1959.

[da Costa, 1974] N. C. A. da Costa. On the theory of inconsistent formal systems. *Notre Dame Journal of Formal Logic*, 15(4):497–510, 1974.

[da Costa, 1993] N. C. A. da Costa. *Sistemas Formais Inconsistentes* (*Inconsistent Formal Systems*, in Portuguese), Habilitation Thesis, 1963. UFPR, Curitiba, 1993. URL = `http://www.cfh.ufsc.br/~nel/historia_logica/sistemas_formais.htm`.

[da Silva *et al.*, 1999] J. J. da Silva, I. M. L. D'Ottaviano, and A. M. Sette. Translations between logics. In X. Caicedo and C. H. Montenegro, editors, *Models, Algebras and Proofs*, pages 435–448, New York, 1999. Marcel Dekker.

[de Amo *et al.*, 2002] S. de Amo, W. A. Carnielli, and J. Marcos. A logical framework for integrating inconsistent information in multiple databases. In T. Eiter and K.-D. Schewe, editors, *Proceedings of the 2nd Symposium on Foundations of Information and Knowledge Systems (FOIKS 2002)*, volume 2284 of *Lecture Notes in Computer Science*, pages 67–84, Berlin, 2002. Springer-Verlag. Preprint available at *CLE e-Prints*, 1(9), 2001. URL = `http://www.cle.unicamp.br/e-prints/abstract_9.htm`.

[D'Ottaviano and da Costa, 1970] I. M. L. D'Ottaviano and N. C. A. da Costa. Sur un problème de Jaśkowski. *Comptes Rendus de l'Académie de Sciences de Paris (A-B)*, 270:1349–1353, 1970.

[D'Ottaviano, 1990] I. M. L. D'Ottaviano. On the development of paraconsistent logic and da Costa's work. *The Journal of Non-Classical Logic*, 7(1/2):89–152, 1990.

[Došen, 1986] K. Došen. Negation as a modal operator. *Reports on Mathematical Logic*, 20:15–28, 1986.

[Epstein, 2000] R. L. Epstein. *Propositional Logics: The semantic foundations of logic,* with the assistance and collaboration of W.A. Carnielli, I.M.L. D'Ottaviano, S. Krajewski, and R.D. Maddux. Wadsworth-Thomson Learning, 2^{nd} edition, 2000.

[Jaśkowski, 1948] S. Jaśkowski. Rachunek zdań dla systemów dedukcyjnych sprzecznych. *Studia Societatis Scientiarun Torunesis, Sectio A*, I(5):57–77, 1948. Translated as 'A propositional calculus for inconsistent deductive systems' in *Logic and Logic Philosophy*, 7:35–56, 1999, Proceedings of the Stanislaw Jaśkowski's Memorial Symposium, held in Toruń, Poland, July 1998.

[Jaśkowski, 1949] S. Jaśkowski. O koniunkcji dyskusyjnej w rachunku zdań dla systemów dedukcyjnych sprzecznych. *Studia Societatis Scientiarun Torunesis, Sectio A*, I(8):171–172, 1949. Translated as 'On the discussive conjunction in the propositional calculus for inconsistent deductive systems' in *Logic and Logic Philosophy*, 7:57–59, 1999, Proceedings of the Stanislaw Jaśkowski's Memorial Symposium, held in Toruń, Poland, July 1998.

[Johánsson, 1936] I. Johánsson. Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus. *Compositio Mathematica*, 4(1):119–136, 1936.

[Kolmogorov, 1967] A. N. Kolmogorov. On the principle of excluded middle. In Van Heijenoort, editor, *From Frege to Gödel*, pages 414–437, Cambridge, 1967. Harvard University Press. Translation from the Russian original (1925).

[Lenzen, 1998] W. Lenzen. Necessary conditions for negation-operators (with particular applications to paraconsistent negation). In P. Besnard and A. Hunter, editors, *Reasoning with Actual and Potential Contradictions*, pages 211–239, Dordrecht, 1998. Kluwer.

[Lewin *et al.*, 1990] R. A. Lewin, I. F. Mikenberg, and M. G. Schwarze. Algebraization of paraconsistent logic $\mathbf{P}^1$. *The Journal of Non-Classical Logic*, 7(1/2):79–88, 1990.

[Loparić and da Costa, 1984] Andrea Loparić and Newton C. A. da Costa. Paraconsistency, paracompleteness, and valuations. *Logique et Analyse (N.S.)*, 106:119–131, 1984.

[Marcos, 1999] J. Marcos. Semânticas de Traduções Possíveis (Possible-Translations Semantics, in Portuguese). Master's thesis, State University of Campinas (Unicamp), Campinas, 1999. URL = `http://www.cle.unicamp.br/pub/thesis/J.Marcos/`.

[Marcos, 2000] J. Marcos. 8K solutions and semi-solutions to a problem of da Costa. Draft, 2000.

[Marcos, 2001] J. Marcos. On a problem of da Costa. *CLE e-Prints*, 1(8), 2001. To appear in *Logica Trianguli*. Preprint available at
URL = `http://www.cle.unicamp.br/e-prints/abstract_8.htm`.

[Marcos, 2004a] J. Marcos. Nearly every normal modal logic is paranormal. Technical report, CLC, Department of Mathematics, Instituto Superior Técnico, 1049-001 Lisbon, PT, 2004. Submitted for publication. Preprint available at
URL = `http://wslc.math.ist.utl.pt/ftp/pub/MarcosJ/04-M-Paranormal.pdf`.

[Marcos, 2004b] J. Marcos. On negation: Pure local rules. *Journal of Applied Logic*, 2004. In print. Preprint available at *CLE e-Prints*, 4(4), 2004.
URL = `http://www.cle.unicamp.br/e-prints/vol_4,n_4,2004.html`.

[Marcos, 2004c] J. Marcos. Possible-translations semantics. In W. A. Carnielli, F. M. Dionísio, and P. Mateus, editors, *Proceedings of the Workshop on Combination of Logics: Theory and applications* (CombLog'04), held in Lisbon, PT, 28–30 July 2004, pages 119–128. Departamento de Matemática, Instituto Superior Técnico, 2004. Preprint available at
URL = `http://wslc.math.ist.utl.pt/ftp/pub/MarcosJ/04-M-pts.pdf`.

[Marcos, 2004d] J. Marcos. Possible-translations semantics for some weak classically-based paraconsistent logics. Research report, CLC, Department of Mathematics, Instituto Superior Técnico, 1049-001 Lisbon, PT, 2004.
URL = `http://www.cs.math.ist.utl.pt/ftp/pub/MarcosJ/04-M-PTS4swcbPL.pdf`.

[Marcos, 2004e] João Marcos. Logics of essence and accident. Technical report, CLC, Department of Mathematics, Instituto Superior Técnico, 1049-001 Lisbon, PT, 2004. Submitted for publication. Preprint available at
URL = `http://www.cs.math.ist.utl.pt/ftp/pub/MarcosJ/04-M-LEA.pdf`.

[Mendelson, 1997] E. Mendelson. *Introduction to Mathematical Logic*. International Thomson Publishing, 4^{th} edition, 1997.

[Miller, 2000] D. Miller. Paraconsistent logic for falsificationists. In *Proceedings of the First Workshop on Logic and Language (Universidad de Sevilla)*, pages 197–204, Sevilla, 2000. Editorial Kronos S.A.

[Mortensen, 1995] C. Mortensen. *Inconsistent Mathematics,* with contributions by P. Lavers, W. James, and J. Cole. Kluwer, Dordrecht, 1995.

[Nelson, 1959] D. Nelson. Negation and separation of concepts in constructive systems. In A. Heyting, editor, *Constructivity in Mathematics,* Proceedings of the Colloquium held in Amsterdam, NL, 1957, Studies in Logic and the Foundations of Mathematics, pages 208–225, Amsterdam, 1959. North-Holland.

[Odintsov, 2004] S. Odintsov. On the structure of paraconsistent extensions of Johánsson's logic. *Journal of Applied Logic*, 2004. In print.

[Popper, 1948] K. R. Popper. On the theory of deduction. Part I and II. *Indagationes Mathematicae*, 10:173–183/322–331, 1948.

[Popper, 1959] K. R. Popper. *The Logic of Scientific Discovery.* Hutchinson & Co., Ltd., London, 1959. (English translation of *Logik der Forschung*, Julius Springer Verlag, Vienna, 1936).

[Popper, 1989] K. R. Popper. *Conjectures and Refutations. The Growth of Scientific Knowledge.* Routledge & Kegan Paul, London, 5^{th} edition, 1989.

[Priest and Routley, 1989] G. Priest and R. Routley. Systems of paraconsistent logic. In *[Priest* et al.*, 1989]*, pages 151–186, 1989.

[Priest *et al.*, 1989] G. Priest, R. Routley, and J. Norman (Editors). *Paraconsistent Logic: Essays on the inconsistent.* Philosophia Verlag, Munich, 1989.

[Priest, 1979] G. Priest. The logic of paradox. *Journal of Philosophical Logic*, 8(2):219–241, 1979.

[Priest, 2002] G. Priest. Paraconsistent logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, 2^{nd} Edition*, volume 6, pages 259–358. Kluwer Academic Publishers, 2002.

[Routley and Meyer, 1976] R. Routley and R. K. Meyer. Dialectical logic, classical logic and the consistence of the world. *Studies in Soviet Thought*, 16:1–25, 1976.

[Schütte, 1960] K. Schütte. *Beweistheorie.* Springer-Verlag, Berlin, 1960.

[Seoane and de Alcantara, 1991] J. Seoane and L. P. de Alcantara. On da Costa algebras. *The Journal of Non-Classical Logic*, 8(2):41–66, 1991.

[Sette, 1973] A. M. Sette. On the propositional calculus $\mathbf{P}^1$. *Mathematica Japonicae*, 18(13):173–180, 1973.

[Shramko, 2004] Y. Shramko. The logic of scientific research. Dresden Preprints in Theoretical Philosophy and Philosophical Logic 7, 2004. 13 pages.

[Slater, 1995] B. H. Slater. Paraconsistent logics? *Journal of Philosophical Logic*, 24(4):451–454, 1995.

[Sylvan, 1990] R. Sylvan. Variations on da Costa **C**-systems and dual-intuitionistic logics. I. Analyses of C_ω and CC_ω. *Studia Logica*, 49(1):47–65, 1990.

[Urbas, 1989] I. Urbas. Paraconsistency and the **C**-systems of da Costa. *Notre Dame Journal of Formal Logic*, 30(4):583–597, 1989.

[Urbas, 1990] I. Urbas. Paraconsistency. *Studies in Soviet Thought*, 39:343–354, 1990.

[Urbas, 1996] I. Urbas. Dual-intuitionistic logic. *Notre Dame Journal of Formal Logic*, 37(3):440–451, 1996.

[Vakarelov, 1989] D. Vakarelov. Consistency, completeness and negation. In *[Priest* et al.*, 1989]*, pages 328–363, 1989.

[Wójcicki, 1988] R. Wójcicki. *Theory of Logical Calculi.* Synthese Library. Kluwer Academic Publishers, 1988.