POSSIBLE-TRANSLATIONS ALGEBRAIZABILITY

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ABSTRACT: The interest of investigating combinations of logics has a philosophical side, a purely logic-theoretical side and promising applicational interests. This paper concentrates on the logic-theoretical side and studies some general methods for combination and decomposition, taking into account that the general idea of combining logics requires methods not only for building new logics, but also for breaking logics into families of logics with lower semantical complexity. The possible-translations semantics, introduced in [8] and subsequently refined, is a particularly apt tool for analyzing and providing semantical meaning and algebraic contents to certain complex logics as paraconsistent logics, and this paper characterizes the possible-translations semantics (**PTS**) and the concept of algebraizability via a **PTS**s in categorial terms, showing that the product of finitely-algebraizable (or Blok-Pigozzi algebraizable) propositional logics is also finitely-algebraizable, under certain very reasonable conditions. Examples and some research directions are discussed.

Keywords: possible-translations semantics; finitely algebraizable logics; categories of logical systems.

1 ALGEBRAIZING, SPLITTING AND INTERPRETING: THE SCOPE OF POSSIBLE-TRANSLATIONS SEMANTICS

The emergent area of combinations of logics can be approached from three sides: from a philosophical perspective, from a purely logic-theoretical side and from the point of view of applications. From the philosophical side, it is connected with a pluralist view on logic and its consequences; from the side of applications, many tasks in knowledge representation or software engineering involve combining knowledge, time and spatial reasoning, among other logic dimensions. But the purely logic-theoretical side offers real challenges; in particular, the general idea of combining logics involves not only the construction of new logics starting from simpler components, but also the concept of breaking logics into families of logics with lower semantical complexity. In a broad view, the activity of combining logics also includes, besides synthesizing logical systems by means of compositional procedures (as, for instance, fibring), the contrary direction of breaking down a logic in terms of other (less complex) logics. Such a reversing procedure is called *splitting*, as opposite to the process of *splicing* (cf. [10]).

This paper is an expanded version of [4], and our intention here is to present in all details a categorial characterization of the process of splitting logics called *possible-translations semantics* (**PTS**s), (cf. [8] and [9]). Although such semantics is adequate for providing interpretation to several non-standard logics (as paraconsistent and to many-valued logics, for instance), and were conceived with these aims, they also constitute a widely general method for splitting logics.

On the other hand, by combining algebraic techniques with a **PTS**s, one obtains a new notion of algebraizability (cf. [5]) extending the method of finitely algebraizable logics due to W. Blok and D. Pigozzi in [2]. This extended notion of algebraization offers a solution to the question of obtaining an exact algebraic counterpart to certain logics which are not amenable to the method of Blok and Pigozzi, as it is the case of various paraconsistent logics.

The purposes of this paper are three: first, to obtain a representation theorem for a possible-translations semantics (**PTS**s); second, to characterize the concept of algebraizability via a **PTS**s in categorial terms; and third, to prove a Finiteness Preservation result (Lemma 4.4), showing that, under certain conditions, the product of finitely-algebraizable (or Blok-Pigozzi algebraizable) propositional logics is also finitely-algebraizable.

Our intention is not only to present such results as if they came out of the blue, but to reconstruct the main steps of our investigations devoted to reach the adequate definitions and the precise constraints under which some basic lemmas hold, aiming to leave a method for the benefit of the reader, so that other topics on the more technical side of logic could be treated an analogous way.

This is achieved by defining the categories **PS** of propositional languages, and **CR** of propositional logics (defined through consequence relations), and showing that they are closed under arbitrary products. This permits to specify the category **ACR** of algebraizable logics as a subcategory of **CR**. Examples and some research directions are also discussed.

2 PROPOSITIONAL LANGUAGES IN A CATEGORIAL CLOTHING

This long section gives, in a good number of details, several ingredients that aim at characterizing propositional languages in general in a categorial clothing. This, instead of making them costly, makes propositional languages a very useful resource, as we shall make clear.

Definition 2.1. A signature is a denumerable family $\Sigma = \{\Sigma_k\}_{k\in\omega}$, where each Σ_k is a set (of connectives of arity k) such that $\Sigma_k \cap \Sigma_n = \emptyset$ if $k \neq n$. The domain of Σ is the set $|\Sigma| = \bigcup_{n\in\omega} \Sigma_n$. We fix a denumerable set $\mathcal{V} = \{p_k : k \in \omega, k \geq 1\}$ of *(propositional) variables* such that $p_k \neq p_n$ whenever $k \neq n$. The *(propositional)* *language generated by* Σ, denoted by $L(\Sigma)$, is the algebra of type Σ freely generated by $\mathcal{V}$. Elements of $L(\Sigma)$ are called *formulas*. For every $n \geq 0$ let

$$L(\Sigma)[n] = \{\varphi \in L(\Sigma) : \text{the variables occurring in } \varphi \text{ are exactly } p_1, \ldots, p_n\}.$$

We write $\varphi(p_1, \ldots, p_n)$ to indicate that the propositional variables occurring in φ are among $p_1, \ldots, p_n$. The notion of complexity $l(\varphi)$ of a formula φ is defined as usual, stipulating that $l(\varphi) = 1$ whenever $\varphi \in \mathcal{V} \cup \Sigma_0$ and $l(c(\alpha_1, \ldots, \alpha_n)) = 1 + l(\alpha_1) + l(\alpha_2) + \ldots + l(\alpha_n)$, if $c \in \Sigma_n$.

Definition 2.2. Let Σ be a signature. A substitution on $L(\Sigma)$ is a function $\sigma{:}\mathcal{V} \to L(\Sigma)$. We denote by $\widehat{\sigma}$ the unique extension of σ to an endomorphism $\widehat{\sigma} : L(\Sigma) \to L(\Sigma)$, such that

(a) $\widehat{\sigma}(p) = \sigma(p)$, if $p \in \mathcal{V}$;

(b) $\widehat{\sigma}(c) = c$, if $c \in \Sigma_0$;

(c) $\widehat{\sigma}(c(\alpha_1, \ldots, \alpha_n)) = c(\widehat{\sigma}(\alpha_1), \ldots, \widehat{\sigma}(\alpha_n))$, if $c \in \Sigma_n$ and $\alpha_1, \ldots, \alpha_n \in L(\Sigma)$.

Given substitutions $\sigma, \sigma' : \mathcal{V} \to L(\Sigma)$, then their *product* $\sigma'\sigma$ is the substitution $\widehat{\sigma'} \circ \sigma$.

Notation 2.3. Given $\varphi(p_1, \ldots, p_n)$ and σ such that $\sigma(p_i) = \alpha_i$ $(i = 1, \ldots, n)$, then $\widehat{\sigma}(\varphi)$ will be denoted by $\varphi(\alpha_1, \ldots, \alpha_n)$

Definition 2.4. Let Σ and Σ' be signatures. A *signature morphism* f from Σ to Σ', denoted $\Sigma \xrightarrow{f} \Sigma'$, is a map $f : |\Sigma| \to L(\Sigma')$ such that if $c \in \Sigma_n$, then $f(c) \in L(\Sigma')[n]$.

Given a signature morphism $\Sigma \xrightarrow{f} \Sigma'$, a map $\widehat{f} : L(\Sigma) \to L(\Sigma')$ can be defined in a natural way:

1. $\widehat{f}(p) = p$ if $p \in \mathcal{V}$;
2. $\widehat{f}(c) = f(c)$ if $c \in \Sigma_0$;
3. $\widehat{f}(c(\alpha_1, \ldots, \alpha_n)) = f(c)(\widehat{f}(\alpha_1), \ldots, \widehat{f}(\alpha_n))$ if $c \in \Sigma_n$ and $\alpha_1, \ldots, \alpha_n \in L(\Sigma)$.

Lemma 2.5. The extension $\widehat{f}$ of f is unique.

Proof. Suppose that $\widehat{g}$ is another extension of f; we shall show, by induction on complexity of formulas, that $\widehat{f} = \widehat{g}$.

- If φ is a propositional letter p, i.e, $p \in \mathcal{V}$:
  $\widehat{f}(\varphi) \overset{\varphi \equiv p}{=} \widehat{f}(p) = p = \widehat{g}(p) \overset{\varphi \equiv p}{=} \widehat{g}(\varphi)$.
- If φ is a constant c, i.e, $c \in \Sigma_0$:
  $\widehat{f}(\varphi) \overset{\varphi \equiv c}{=} \widehat{f}(c) = f(c) = \widehat{g}(c) \overset{\varphi \equiv c}{=} \widehat{g}(\varphi)$.

- If φ is a formula of the form $c(\alpha_1, \ldots, \alpha_n)$, for, $c \in \Sigma_n$:

  Suppose, by induction hypothesis, that the result is valid for every formula α, such that $l(\alpha) < l(\varphi)$; then:

  $\widehat{f}(\varphi) = \widehat{f}(c(\alpha_1, \ldots, \alpha_n)) = f(c)(\widehat{f}(\alpha_1), \ldots, \widehat{f}(\alpha_n)) \overset{\text{Ind.Hyp.}}{=}$ $f(c)(\widehat{g}(\alpha_1), \ldots, \widehat{g}(\alpha_n)) = \widehat{g}(c(\alpha_1, \ldots, \alpha_n)) = \widehat{g}(\varphi)$.

■

Definition 2.6. Let $\Sigma \overset{f}{\to} \Sigma'$ and $\Sigma' \overset{g}{\to} \Sigma''$ be signature morphisms. The *composition* $g \centerdot f$ of f and g is the signature morphism $\Sigma \overset{g \centerdot f}{\to} \Sigma''$ given by the map $\widehat{g} \circ f : |\Sigma| \to L(\Sigma'')$.

The following technical results will be useful.

Lemma 2.7. $\widehat{\sigma'\sigma} = \widehat{\sigma'} \circ \widehat{\sigma}$.

Proof. By induction on the complexity of φ.

- If φ is a variable p, i.e, $p \in \mathcal{V}$:

  $\widehat{\sigma'\sigma}(\varphi) \overset{\varphi = p}{=} \widehat{\sigma'\sigma}(p) \overset{\text{Def.2.2(a)}}{=} \sigma'\sigma(p) \overset{\text{Def.2.2}}{=} (\widehat{\sigma'} \circ \sigma)(p) = \widehat{\sigma'}(\sigma(p)) \overset{\text{Def.2.2(a)}}{=}$ $\widehat{\sigma'}(\widehat{\sigma}(p)) \overset{\text{Def.2.2}}{=} (\widehat{\sigma'} \circ \widehat{\sigma})(p) \overset{\varphi = p}{=} (\widehat{\sigma'} \circ \widehat{\sigma})(\varphi)$.

- If φ is a constant c, i.e, $c \in \Sigma_0$:

  $\widehat{\sigma'\sigma}(\varphi) \overset{\varphi = c}{=} \widehat{\sigma'\sigma}(c) \overset{\text{Def.2.2(b)}}{=} c \overset{\text{Def.2.2(b)}}{=} \widehat{\sigma'}(c) \overset{\text{Def.2.2(b)}}{=} \widehat{\sigma'}(\widehat{\sigma}(c)) = (\widehat{\sigma'} \circ \widehat{\sigma})(c) \overset{\varphi = c}{=}$ $(\widehat{\sigma'} \circ \widehat{\sigma})(\varphi)$.

- If φ is a formula $c(\alpha_1, \ldots, \alpha_n)$, for $c \in \Sigma_n$:

  $\widehat{\sigma'\sigma}(\varphi) = \widehat{\sigma'\sigma}(c(\alpha_1, \ldots, \alpha_n)) \overset{\text{Def.2.2(c)}}{=} c(\widehat{\sigma'\sigma}(\alpha_1), \ldots, \widehat{\sigma'\sigma}(\alpha_n)) \overset{\text{Ind.Hyp.}}{=}$ $c(\widehat{\sigma'}(\widehat{\sigma}(\alpha_1)), \ldots, \widehat{\sigma'}(\widehat{\sigma}(\alpha_n))) \overset{\text{Def.2.2(c)}}{=} \widehat{\sigma'}(c(\widehat{\sigma}(\alpha_1), \ldots, \widehat{\sigma}(\alpha_n))) \overset{\text{Def.2.2(c)}}{=}$ $\widehat{\sigma'}(\widehat{\sigma}(c(\alpha_1, \ldots, \alpha_n))) = (\widehat{\sigma'} \circ \widehat{\sigma})(c(\alpha_1, \ldots, \alpha_n)) = (\widehat{\sigma'} \circ \widehat{\sigma})(\varphi)$.

■

Lemma 2.8. Let $\varphi(p_1, \ldots, p_n)$ be a formula, and let $\sigma, \sigma' : \mathcal{V} \to L(\Sigma)$ be substitutions such that: $\sigma(p_i) = \alpha_i$ $(i = 1, \ldots, n)$. Then $\widehat{\sigma'}(\varphi(\alpha_1, \ldots, \alpha_n)) = \varphi(\widehat{\sigma'}(\alpha_1), \ldots, \widehat{\sigma'}(\alpha_n))$.

Proof.

$$
\begin{aligned}
\widehat{\sigma'}(\varphi(\alpha_1, \ldots, \alpha_n)) &\overset{\text{cf. 2.3}}{=} \widehat{\sigma'}(\widehat{\sigma}(\varphi(p_1, \ldots, p_n))) \\
&= (\widehat{\sigma'} \circ \widehat{\sigma})(\varphi(p_1, \ldots, p_n)) \\
&\overset{\text{cf. 2.7}}{=} \widehat{\sigma'\sigma}(\varphi(p_1, \ldots, p_n)) \\
&\overset{\text{cf. 2.3}}{=} \varphi(\widehat{\sigma'\sigma}(p_1), \ldots, \widehat{\sigma'\sigma}(p_n)) \\
&\overset{\text{cf. 2.7}}{=} \varphi((\widehat{\sigma'} \circ \widehat{\sigma})(p_1), \ldots, (\widehat{\sigma'} \circ \widehat{\sigma})(p_n)) \\
&= \varphi(\widehat{\sigma'}(\widehat{\sigma}(p_1)), \ldots, \widehat{\sigma'}(\widehat{\sigma}(p_n))) \\
&= \varphi(\widehat{\sigma'}(\sigma(p_1)), \ldots, \widehat{\sigma'}(\sigma(p_n))) \\
&\overset{\sigma(p_i) = \alpha_i}{=} \varphi(\widehat{\sigma'}(\alpha_1), \ldots, \widehat{\sigma'}(\alpha_n)).
\end{aligned}
$$

■

Lemma 2.9. Let $\varphi = \varphi(p_1, \ldots, p_n)$ in $L(\Sigma)$, $\alpha_1, \ldots, \alpha_n \in L(\Sigma)$ and $\Sigma \xrightarrow{f} \Sigma'$ a signature morphism. Then $\widehat{f}(\varphi(\alpha_1, \ldots, \alpha_n)) = \widehat{f}(\varphi)(\widehat{f}(\alpha_1), \ldots, \widehat{f}(\alpha_n))$.

Proof. By induction on the complexity $l(\varphi)$ of φ. Let us denote the sequence $\alpha_1, \ldots, \alpha_n$ by $\vec{\alpha}$.

- If φ is a propositional variable $p_i \in \mathcal{V}$;

$$\begin{array}{rcl} \widehat{f}(\varphi(\vec{\alpha})) & \stackrel{\varphi \,\equiv\, p_i}{=} & \widehat{f}(p_i(\vec{\alpha})) \\ & = & \widehat{f}(\alpha_i) \\ & = & p_i(\widehat{f}(\alpha_1), \ldots, \widehat{f}(\alpha_n)) \\ & = & \widehat{f}(p_i)(\widehat{f}(\alpha_1), \ldots, \widehat{f}(\alpha_n)) \\ & \stackrel{\varphi \,\equiv\, p_i}{=} & \widehat{f}(\varphi)(\widehat{f}(\alpha_1), \ldots, \widehat{f}(\alpha_n)) \end{array}$$

- If φ is a constant $c \in \Sigma_0$, Then $\varphi(\vec{\alpha}) = c$, hence:

$$\begin{array}{rcl} \widehat{f}(\varphi(\vec{\alpha})) & \stackrel{\varphi \,\equiv\, c}{=} & \widehat{f}(c(\vec{\alpha})) \\ & = & f(c)(\widehat{f}(\alpha_1), \ldots, \widehat{f}(\alpha_n)) \\ & = & \widehat{f}(c)(\widehat{f}(\alpha_1), \ldots, \widehat{f}(\alpha_n)) \\ & \stackrel{\varphi \,\equiv\, c}{=} & \widehat{f}(\varphi)(\widehat{f}(\alpha_1), \ldots, \widehat{f}(\alpha_n)) \end{array}$$

- If $\varphi = c(\beta_1, \ldots, \beta_k)$, with $\beta_i = \beta_i(p_1, \ldots, p_n)$ $(i = 1, \cdots, k)$, then $\varphi(\vec{\alpha}) = c(\beta_1(\vec{\alpha}), \ldots, \beta_k(\vec{\alpha}))$, hence:

$$\begin{array}{rcl} \widehat{f}(\varphi(\vec{\alpha})) & = & \widehat{f}(c(\beta_1(\vec{\alpha}), \ldots, \beta_k(\vec{\alpha}))) \\ & = & f(c)(\widehat{f}(\beta_1(\vec{\alpha})), \ldots, \widehat{f}(\beta_k(\vec{\alpha}))) \\ & \stackrel{\text{Ind.Hyp.}}{=} & f(c)(\widehat{f}(\beta_1)(\widehat{f}(\alpha_1), ..., \widehat{f}(\alpha_n)), \ldots, \widehat{f}(\beta_k)(\widehat{f}(\alpha_1), ..., \widehat{f}(\alpha_n))) \\ & = & f(c)(\widehat{f}(\beta_1), \ldots, \widehat{f}(\beta_k))(\widehat{f}(\alpha_1), \ldots, \widehat{f}(\alpha_n)) \\ & = & \widehat{f}(c(\beta_1, \ldots, \beta_k))(\widehat{f}(\alpha_1), \ldots, \widehat{f}(\alpha_n)) \\ & = & \widehat{f}(\varphi)(\widehat{f}(\alpha_1), \ldots, \widehat{f}(\alpha_n)). \end{array}$$

■

Lemma 2.10. Let $\Sigma \xrightarrow{f} \Sigma'$ and $\Sigma' \xrightarrow{g} \Sigma''$ be signature morphisms. Then $\widehat{g \centerdot f} = \widehat{g} \circ \widehat{f}$.

Proof. By induction on the complexity of $l(\varphi)$, for $\varphi \in L(\Sigma)$, we prove that $\widehat{g \centerdot f}(\varphi) = \widehat{g} \circ \widehat{f}(\varphi)$.

- If $\varphi = p$, $p \in \mathcal{V}$, then:
  $\widehat{g \centerdot f}(\varphi) \stackrel{\varphi \,\equiv\, p}{=} \widehat{g \centerdot f}(p) = p = \widehat{g}(p) = \widehat{g}(\widehat{f}(p)) = \widehat{g} \circ \widehat{f}(p) \stackrel{\varphi \,\equiv\, p}{=} \widehat{g} \circ \widehat{f}(\varphi)$.

- If $\varphi = c$, for $c \in \Sigma_0$, then:
  $\widehat{g \centerdot f}(\varphi) \stackrel{\varphi \,\equiv\, c}{=} \widehat{g \centerdot f}(c) = g \centerdot f(c) \stackrel{\text{Def.2.6}}{=} \widehat{g} \circ f(c) = \widehat{g}(f(c)) = \widehat{g}(\widehat{f}(c)) = \widehat{g} \circ \widehat{f}(c) \stackrel{\varphi \,\equiv\, c}{=} \widehat{g} \circ \widehat{f}(\varphi)$.

- If $\varphi = c(\alpha_1, \ldots, \alpha_n)$ and $c \in \Sigma_n$, then:

$$
\begin{array}{rcl}
\widehat{g \centerdot f}(\varphi) & = & \widehat{g \centerdot f}(c(\alpha_1, \ldots, \alpha_n)) \\
& = & g \centerdot f(c)(\widehat{g \centerdot f}(\alpha_1), \cdots, \widehat{g \centerdot f}(\alpha_n)) \\
& \stackrel{\text{Def.2.6}}{=} & \widehat{g} \circ f(c)(\widehat{g \centerdot f}(\alpha_1), \ldots, \widehat{g \centerdot f}(\alpha_n)) \\
& = & \widehat{g}(f(c))(\widehat{g \centerdot f}(\alpha_1), \ldots, \widehat{g \centerdot f}(\alpha_n)) \\
& \stackrel{\text{Ind.Hyp.}}{=} & \widehat{g}(f(c))(\widehat{g} \circ \widehat{f}(\alpha_1), \cdots, \widehat{g} \circ \widehat{f}(\alpha_n)) \\
& = & \widehat{g}(f(c))(\widehat{g}(\widehat{f}(\alpha_1)), \cdots, \widehat{g}(\widehat{f}(\alpha_n))) \\
& \stackrel{\text{cf. 2.9}}{=} & \widehat{g}(f(c)(\widehat{f}(\alpha_1), \cdots, \widehat{f}(\alpha_n))) \\
& = & \widehat{g}(\widehat{f}(c(\alpha_1, \ldots, \alpha_n))) \\
& = & \widehat{g}(\widehat{f}(\varphi)) \\
& = & \widehat{g} \circ \widehat{f}(\varphi).
\end{array}
$$

■

Lemma 2.11. Let $\Sigma \xrightarrow{f} \Sigma'$ be a signature morphism, and let $\sigma{:}\mathcal{V} \to L(\Sigma)$ be a substitution over Σ. Then there is a substitution $\sigma'{:}\mathcal{V} \to L(\Sigma')$ over Σ' such that $\widehat{f} \circ \widehat{\sigma} = \widehat{\sigma'} \circ \widehat{f}$.

Proof. Define $\sigma'(p) = \widehat{f}(\sigma(p))$ for every $p \in \mathcal{V}$. By induction on the complexity $l(\alpha)$ of α we can prove that $\widehat{f}(\widehat{\sigma}(\alpha)) = \widehat{\sigma'}(\widehat{f}(\alpha))$ for every α.

- If $\alpha = p \in \mathcal{V}$, then:
  $\widehat{f}(\widehat{\sigma}(\alpha)) \stackrel{\alpha = p}{=} \widehat{f}(\widehat{\sigma}(p)) \stackrel{\text{Def.2.2(a)}}{=} \widehat{f}(\sigma(p)) = \sigma'(p) \stackrel{\text{Def.2.2(a)}}{=} \widehat{\sigma'}(p) = \widehat{\sigma'}(\widehat{f}(p)) \stackrel{\alpha = p}{=} \widehat{\sigma'}(\widehat{f}(\alpha))$

- If $\alpha = c \in \Sigma_0$, then:
  $\widehat{f}(\widehat{\sigma}(\alpha)) \stackrel{\alpha = c}{=} \widehat{f}(\widehat{\sigma}(c)) \stackrel{\text{Def.2.2(b)}}{=} \widehat{f}(c) \stackrel{\text{Def.2.2(b)}}{=} \widehat{\sigma'}(\widehat{f}(c)) \stackrel{\alpha = c}{=} \widehat{\sigma'}(\widehat{f}(\alpha))$, because $\widehat{f}(c) \in L(\Sigma')[0]$.

- If $\alpha = c(\alpha_1, \ldots, \alpha_k)$ for $c \in \Sigma_k$, then:

$$
\begin{array}{rcl}
\widehat{f}(\widehat{\sigma}(\alpha)) & = & \widehat{f}(\widehat{\sigma}(c(\alpha_1, \ldots, \alpha_k))) \\
& \stackrel{\text{Def.2.2 (c)}}{=} & \widehat{f}(c(\widehat{\sigma}(\alpha_1), \ldots, \widehat{\sigma}(\alpha_k))) \\
& = & f(c)(\widehat{f}(\widehat{\sigma}(\alpha_1)), \ldots, \widehat{f}(\widehat{\sigma}(\alpha_k))) \\
& \stackrel{\text{Ind.Hyp.}}{=} & f(c)(\widehat{\sigma'}(\widehat{f}(\alpha_1)), \ldots, \widehat{\sigma'}(\widehat{f}(\alpha_k))) \\
& \stackrel{\text{cf. 2.8}}{=} & \widehat{\sigma'}(f(c)(\widehat{f}(\alpha_1), \ldots, \widehat{f}(\alpha_k))) \\
& = & \widehat{\sigma'}(\widehat{f}(c(\alpha_1, \ldots, \alpha_k))) \\
& = & \widehat{\sigma'}(\widehat{f}(\alpha))
\end{array}
$$

■

Definition 2.12. The category **PS** of (propositional) languages is defined as follows:

- Objects: Propositional signatures (cf. Definition 2.1);
- Morphisms: Signature morphisms (cf. Definition 2.4);
- Composition: As in Definition 2.6;
- Identity morphisms: For every signature Σ the identity morphism $\Sigma \xrightarrow{id_\Sigma} \Sigma$ is defined by:
  $id_\Sigma(c) = c$ (for $c \in \Sigma_0$) and
  $id_\Sigma(c) = c(p_1, \ldots, p_n)$ (for $c \in \Sigma_n$, $n \geq 1$).

Proposition 2.13. **PS** is, in fact, a category.

Proof. Let $\Sigma \xrightarrow{f} \Sigma' \xrightarrow{g} \Sigma'' \xrightarrow{h} \Sigma'''$ be signature morphisms. We must show that the composition $\centerdot$ is associative.

$$h \centerdot (g \centerdot f) \overset{\text{Def.2.6}}{=} h \centerdot (\widehat{g} \circ f) \overset{\text{Def.2.6}}{=} \widehat{h} \circ (\widehat{g} \circ f) = (\widehat{h} \circ \widehat{g}) \circ f \overset{\text{cf. 2.10}}{=} \widehat{h \centerdot g} \circ f \overset{\text{Def.2.6}}{=} (h \centerdot g) \centerdot f.$$

Finally, it rests to show that the signature morphisms $\Sigma' \xrightarrow{f} \Sigma \xrightarrow{g} \Sigma''$ verify the following identities: $id_\Sigma \centerdot f = f$ and $g \centerdot id_\Sigma = g$. But it easily follows from Lemma 2.7, Definition 2.6 and Lemma 2.5. ■

Proposition 2.14. The category **PS** has products of arbitrary (small, non-empty) families of objects.

Proof. Let $\mathcal{F} = \{\Sigma^i\}_{i \in I}$ be a family of signatures such that I is a non-empty set. Consider the signature $\Sigma^{\mathcal{F}}$ such that, for every $n \in \omega$,

$$\Sigma^{\mathcal{F}}_n = \{(\varphi_i)_{i \in I} : \varphi_i \in L(\Sigma^i)[n] \text{ for every } i \in I\}.$$

For each $i \in I$, consider the map $\pi_i : |\Sigma^{\mathcal{F}}| \to L(\Sigma^i)$ such that $\pi_i((\varphi_i)_{i \in I}) = \varphi_i$ if $(\varphi_i)_{i \in I} \in \Sigma^{\mathcal{F}}_n$, for $n \in \omega$. Then π_i determines a **PS**-morphism $\Sigma^{\mathcal{F}} \xrightarrow{\pi_i} \Sigma^i$. Consider a signature Σ' together with **PS**-morphisms $\Sigma' \xrightarrow{f_i} \Sigma^i$, for $i \in I$. Let $f : |\Sigma'| \to L(\Sigma^{\mathcal{F}})$ such that $f(c) = (f_i(c))_{i \in I}(p_1, \ldots, p_n)$ if $c \in {\Sigma'}_n$, for $n \in \omega$. Then f defines a **PS**-morphism $\Sigma' \xrightarrow{f} \Sigma^{\mathcal{F}}$ such that $f_i = \pi_i \centerdot f$ for every $i \in I$. If $\Sigma' \xrightarrow{g} \Sigma^{\mathcal{F}}$ is a morphism such that $f_i = \pi_i \centerdot g$ for every $i \in I$ then clearly $g = f$. This proves that $\langle \Sigma^{\mathcal{F}}, \{\pi_i\}_{i \in I} \rangle$ is the product in **PS** of the family $\mathcal{F}$. ■

3 CONSEQUENCE RELATIONS, ALSO IN A CATEGORIAL CLOTHING

In this section we introduce the category **CR** of (propositional) logics defined through consequence relations. As much as for the case of propositional languages, this careful treatment makes the family of logics also a very useful collectivity.

Definition 3.1. A *(propositional) logic* is a pair $\mathcal{L} = \langle \Sigma, \vdash_{\mathcal{L}} \rangle$, where Σ is a signature (cf. Definition 2.1) and $\vdash_{\mathcal{L}}$ is a subset of $\wp(L(\Sigma)) \times L(\Sigma)$ satisfying the following properties, for every $\Gamma \cup \Theta \cup \{\varphi\} \subseteq L(\Sigma)$:

- If $\varphi \in \Gamma$ then $\Gamma \vdash_{\mathcal{L}} \varphi$ (Extensivity);
- If $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\Theta \vdash_{\mathcal{L}} \psi$ for all $\psi \in \Gamma$ then $\Theta \vdash_{\mathcal{L}} \varphi$ (Transitivity);
- If $\Gamma \vdash_{\mathcal{L}} \varphi$ then $\Delta \vdash_{\mathcal{L}} \varphi$ for some finite set $\Delta \subseteq \Gamma$ (Finitariness);
- If $\Gamma \vdash_{\mathcal{L}} \varphi$ then $\widehat{\sigma}(\Gamma) \vdash_{\mathcal{L}} \widehat{\sigma}(\varphi)$ for every substitution σ (Structurality).

The relation $\vdash_{\mathcal{L}}$ is called the *consequence relation* of $\mathcal{L}$.

Note that, because of Extensivity and Transitivity, the following property is satisfied by any consequence relation $\vdash_{\mathcal{L}}$:

- If $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\Gamma \subseteq \Theta$ then $\Theta \vdash_{\mathcal{L}} \varphi$ (Monotonicity).

Definition 3.2. Let $\mathcal{L} = \langle \Sigma, \vdash_{\mathcal{L}} \rangle$ and $\mathcal{L}' = \langle \Sigma', \vdash_{\mathcal{L}'} \rangle$ be logics. A *morphism of logics* f from $\mathcal{L}$ to $\mathcal{L}'$, denoted by $\mathcal{L} \xrightarrow{f} \mathcal{L}'$, is a **PS**-morphism $\Sigma \xrightarrow{f} \Sigma'$ which is a *translation*, that is, it satisfies the following condition, for every $\Gamma \cup \{\varphi\} \subseteq L(\Sigma)$:

$$\Gamma \vdash_{\mathcal{L}} \varphi \ \text{ implies } \ \widehat{f}(\Gamma) \vdash_{\mathcal{L}'} \widehat{f}(\varphi).$$

By defining composition of morphisms and identity morphisms as in **PS** we then obtain a category of (propositional) logics defined through consequence relations, called **CR**. A fundamental property of **CR** is the following:

Proposition 3.3. The category **CR** has products of arbitrary (small, non-empty) families of objects.

Proof. Let $\mathcal{F} = \{\mathcal{L}_i\}_{i \in I}$ be a family of logics, where I is a non-empty set and each $\mathcal{L}_i$ is of form $\langle \Sigma^i, \vdash_{\mathcal{L}_i} \rangle$. Consider the product $\langle \Sigma^{\mathcal{F}}, \{\pi_i\}_{i \in I} \rangle$ of $\{\Sigma^i\}_{i \in I}$ in the category **PS** (cf. Proposition 2.14). Let $\vdash_{\mathcal{F}} \subseteq \wp(L(\Sigma^{\mathcal{F}})) \times L(\Sigma^{\mathcal{F}})$ be the relation defined as follows:

$\Gamma \vdash_{\mathcal{F}} \varphi$ iff there exists a finite set $\Delta \subseteq \Gamma$ such that $\widehat{\pi}_i(\Delta) \vdash_{\mathcal{L}_i} \widehat{\pi}_i(\varphi)$ for every $i \in I$.

Let $\mathcal{L}^{\mathcal{F}} = \langle \Sigma^{\mathcal{F}}, \vdash_{\mathcal{F}} \rangle$. We will show that the pair $\langle \mathcal{L}^{\mathcal{F}}, \{\pi_i\}_{i \in I} \rangle$ is the product in the category **CR** of the family $\mathcal{F}$. Firstly we will see that $\mathcal{L}^{\mathcal{F}}$ is a logic, that is, the relation $\vdash_{\mathcal{F}}$ is a consequence relation (cf. Definition 3.1).

(i) $\vdash_{\mathcal{F}}$ is extensional:
Consider $\Gamma \subseteq L(\Sigma^{\mathcal{F}})$. Let $\varphi \in \Gamma$ and $\Delta = \{\varphi\}$. Then $\varphi \in \Delta$ and Δ is a finite subset of Γ. Since $\langle \Sigma^i, \vdash_{\mathcal{L}_i} \rangle$ is a propositional logic then it satisfies Extensivity, and so $\widehat{\pi}_i(\Delta) \vdash_{\mathcal{L}_i} \widehat{\pi}_i(\varphi)$, for every $i \in I$. But this means that $\Gamma \vdash_{\mathcal{F}} \varphi$.

(ii) $\vdash_{\mathcal{F}}$ is transitive:
Suppose that $\Gamma \vdash_{\mathcal{F}} \varphi$ and $\Theta \vdash_{\mathcal{F}} \psi$ for every $\psi \in \Gamma$. Then there exists a finite subset $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ of Γ such that $\widehat{\pi}_i(\Delta) \vdash_{\mathcal{L}_i} \widehat{\pi}_i(\varphi)$, for every $i \in I$. Let

$1 \le j \le n$. Then $\Theta \vdash_{\mathcal{F}} \gamma_j$ and so there exists a finite subset Δ_j of Θ such that $\widehat{\pi}_i(\Delta_j) \vdash_{\mathcal{L}_i} \widehat{\pi}_i(\gamma_j)$, for every $i \in I$. Let $\Delta' = \bigcup_{j=1}^{n} \Delta_j$. Then Δ' is a finite subset of Θ such that $\widehat{\pi}_i(\Delta') \vdash_{\mathcal{L}_i} \psi$, for every $\psi \in \widehat{\pi}_i(\Delta_j)$, every $j = 1, \ldots, n$ and every $i \in I$, because every $\vdash_{\mathcal{L}_i}$ satisfies Extensivity. Since every $\vdash_{\mathcal{L}_i}$ satisfies Transitivity then $\widehat{\pi}_i(\Delta') \vdash_{\mathcal{L}_i} \widehat{\pi}_i(\gamma_j)$, for every $j = 1, \ldots, n$ and every $i \in I$. Using again the Transitivity of $\vdash_{\mathcal{L}_i}$ we infer that $\widehat{\pi}_i(\Delta') \vdash_{\mathcal{L}_i} \widehat{\pi}_i(\varphi)$, for every $i \in I$. Therefore $\Theta \vdash_{\mathcal{F}} \varphi$.

(iii) $\vdash_{\mathcal{F}}$ is finitary by the very definition.

(iv) $\vdash_{\mathcal{F}}$ is structural:
Consider a set $\Gamma \cup \{\varphi\} \subseteq L(\Sigma^{\mathcal{F}})$ such that $\Gamma \vdash_{\mathcal{F}} \varphi$. Then, there is a finite set $\Delta \subseteq \Gamma$ such that $\widehat{\pi}_i(\Delta) \vdash_{\mathcal{L}_i} \widehat{\pi}_i(\varphi)$ for every $i \in I$. Let $\sigma : \mathcal{V} \to L(\Sigma^{\mathcal{F}})$ be a substitution over $\Sigma^{\mathcal{F}}$. Since every π_i is a **PS**-morphism then, for every $i \in I$, there exists a substitution $\sigma_i : \mathcal{V} \to L(\Sigma^i)$ over Σ^i such that $\widehat{\pi}_i \circ \widehat{\sigma} = \widehat{\sigma}_i \circ \widehat{\pi}_i$, by Lemma 2.11. Since each $\mathcal{L}_i$ satisfies Structurality, then $\widehat{\sigma}_i(\widehat{\pi}_i(\Delta)) \vdash_{\mathcal{L}_i} \widehat{\sigma}_i(\widehat{\pi}_i(\varphi))$, i.e, $\widehat{\pi}_i(\widehat{\sigma}(\Delta)) \vdash_{\mathcal{L}_i} \widehat{\pi}_i(\widehat{\sigma}(\varphi))$ for every $i \in I$, where $\widehat{\sigma}(\Delta)$ is a finite subset of $\widehat{\sigma}(\Gamma)$. Therefore $\widehat{\sigma}(\Gamma) \vdash_{\mathcal{F}} \widehat{\sigma}(\varphi)$ and so $\vdash_{\mathcal{F}}$ satisfies Structurality.

This shows that $\mathcal{L}^{\mathcal{F}}$ is a logic. By the very definition of $\mathcal{L}^{\mathcal{F}}$, each π_i is a **CR**-morphism $\mathcal{L}^{\mathcal{F}} \xrightarrow{\pi_i} \mathcal{L}_i$. Suppose that $\mathcal{L}' = \langle \Sigma', \vdash_{\mathcal{L}'} \rangle$ is a logic and $\mathcal{L}' \xrightarrow{f_i} \mathcal{L}_i$ is a **CR**-morphism, for every $i \in I$. Then there exists a unique **PS**-morphism $\Sigma' \xrightarrow{f} \Sigma^{\mathcal{F}}$ such that, in **PS**, $\pi_i \centerdot f = f_i$, for every $i \in I$, because $\langle \Sigma^{\mathcal{F}}, \{\pi_i\}_{i \in I} \rangle$ is the product of $\{\Sigma^i\}_{i \in I}$ in the category **PS**. Suppose that $\Gamma \cup \{\varphi\} \subseteq L(\Sigma')$ is such that $\Gamma \vdash_{\mathcal{L}'} \varphi$. Since $\mathcal{L}'$ satisfies Finitariness, there exists a finite set $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathcal{L}'} \varphi$. Since each f_i is a **CR**-morphism then $\widehat{f_i}(\Delta) \vdash_{\mathcal{L}_i} \widehat{f_i}(\varphi)$, for every $i \in I$, hence $\widehat{\pi_i \centerdot f}(\Delta) \vdash_{\mathcal{L}_i} \widehat{\pi_i \centerdot f}(\varphi)$. Using Lemma 2.10 we have that $\widehat{\pi}_i \circ \widehat{f}(\Delta) \vdash_{\mathcal{L}_i} \widehat{\pi}_i \circ \widehat{f}(\varphi)$, i.e, $\widehat{\pi}_i(\widehat{f}(\Delta)) \vdash_{\mathcal{L}_i} \widehat{\pi}_i(\widehat{f}(\varphi))$, for every $i \in I$, where $\widehat{f}(\Delta)$ is a finite subset of $\widehat{f}(\Gamma)$. Therefore, by definition of $\vdash_{\mathcal{F}}$ we have that $\widehat{f}(\Gamma) \vdash_{\mathcal{F}} \widehat{f}(\varphi)$ and then f is a **CR**-morphism $\mathcal{L}' \xrightarrow{f} \mathcal{L}^{\mathcal{F}}$ such that, in **CR**, $\pi_i \centerdot f = f_i$, for every $i \in I$. The unicity of f is a consequence of the universal property in the category **PS** of the product $\langle \Sigma^{\mathcal{F}}, \{\pi_i\}_{i \in I} \rangle$. This shows that $\langle \mathcal{L}^{\mathcal{F}}, \{\pi_i\}_{i \in I} \rangle$ is the product in the category **CR** of the family $\mathcal{F}$. ■

4 PRODUCTS OF ALGEBRAIZABLE LOGICS

In this section we prove that, given a (small and non-empty) family $\mathcal{F}$ of finitely algebraizable logics (in the sense of Blok-Pigozzi, cf. [2]) satisfying a bound condition, then the product of $\mathcal{F}$ in **CR** is also an algebraizable logic. This will be used in Section 6.

We begin by briefly recalling the basic definitions of [2].

Definition 4.1. A propositional logic $\mathcal{L} = \langle \Sigma, \vdash_{\mathcal{L}} \rangle$ is *algebraizable* (in the sense of

Blok-Pigozzi) if there exists a finite set $\Delta = \{\Delta^i(p_1, p_2) : 1 \leq i \leq n\}$ of formulas in $L(\Sigma)[2]$, and a finite set $\langle \varepsilon, \delta \rangle = \{\langle \varepsilon^i(p_1), \delta^i(p_1) \rangle : 1 \leq i \leq m\}$ contained in $L(\Sigma)[1] \times L(\Sigma)[1]$ such that, for every $\varphi, \psi, \gamma \in L_\Sigma$:

1. $\vdash_{\mathcal{L}} \varphi \Delta \varphi$;
2. $\varphi \Delta \psi \vdash_{\mathcal{L}} \psi \Delta \varphi$;
3. $\varphi \Delta \psi,\ \psi \Delta \gamma \vdash_{\mathcal{L}} \varphi \Delta \gamma$;
4. $\varphi_1 \Delta \psi_1, \ldots, \varphi_k \Delta \psi_k \vdash_{\mathcal{L}} c(\varphi_1, \ldots, \varphi_k) \Delta c(\psi_1, \ldots, \psi_k)$ for every $c \in \Sigma_k$ and every $\varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_k$ in $L(\Sigma)$;
5. $\varphi \vdash_{\mathcal{L}} \varepsilon(\varphi) \Delta \delta(\varphi)$, and $\varepsilon(\varphi) \Delta \delta(\varphi) \vdash_{\mathcal{L}} \varphi$.

We say that $\langle \Delta, \langle \varepsilon, \delta \rangle \rangle$ is an *algebraizator* for $\mathcal{L}$.

Some remarks on the notation adopted in Definition 4.1: for any $\varphi, \psi \in L(\Sigma)$ then $\varphi \Delta \psi$ denotes the set of formulas $\{\Delta^i(\varphi, \psi) : 1 \leq i \leq n\}$, and $\varepsilon(\varphi) \Delta \delta(\varphi)$ denotes the set $\{\Delta^j(\varepsilon^i(\varphi), \delta^i(\varphi)) : 1 \leq j \leq n \text{ and } 1 \leq i \leq m\}$. Given sets Γ, Θ of formulas then $\Gamma \vdash_{\mathcal{L}} \Theta$ means that $\Gamma \vdash_{\mathcal{L}} \varphi$ for every $\varphi \in \Theta$. Following [16] and [12] we define the category **ACR** of algebraizable logics.

Definition 4.2. The category **ACR** of *algebraizable logics* is the subcategory of **CR** defined as follows:

- Objects: propositional logics $\mathcal{L} = \langle \Sigma, \vdash_{\mathcal{L}} \rangle$ which are algebraizable (cf. Definitions 3.1 and 4.1);
- Morphisms: a morphism $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ is a **CR**-morphism $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ such that, if $\langle \Delta, \langle \varepsilon, \delta \rangle \rangle$ and $\langle \Delta', \langle \varepsilon', \delta' \rangle \rangle$ are algebraizators for $\mathcal{L}$ and $\mathcal{L}'$, respectively, then $p_1 \widehat{f}(\Delta) p_2 \vdash_{\mathcal{L}'} p_1 \Delta' p_2$ and $p_1 \Delta' p_2 \vdash_{\mathcal{L}'} p_1 \widehat{f}(\Delta) p_2$, where $p_1 \widehat{f}(\Delta) p_2$ denotes the set of formulas $\{\widehat{f}(\Delta^i)(p_1, p_2) : 1 \leq i \leq n\}$;
- Composition and identity morphisms: inherited from **CR**.

Remark 4.3. From [2] we obtain the following: let $\langle \Delta, \langle \varepsilon, \delta \rangle \rangle$ and $\langle \Delta', \langle \varepsilon', \delta' \rangle \rangle$ be two algebraizators for a logic $\mathcal{L}$. Then $p_1 \Delta' p_2 \vdash_{\mathcal{L}} p_1 \Delta p_2$ and $p_1 \Delta p_2 \vdash_{\mathcal{L}} p_1 \Delta' p_2$. Therefore, a **CR**-morphism $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ is a **ACR**-morphism iff there are algebraizators $\langle \Delta, \langle \varepsilon, \delta \rangle \rangle$ and $\langle \Delta', \langle \varepsilon', \delta' \rangle \rangle$ for $\mathcal{L}$ and $\mathcal{L}'$, respectively, such that $p_1 \widehat{f}(\Delta) p_2 \vdash_{\mathcal{L}'} p_1 \Delta' p_2$ and $p_1 \Delta' p_2 \vdash_{\mathcal{L}'} p_1 \widehat{f}(\Delta) p_2$ (cf. [16] and [12]).

Now we prove that the product of a family of algebraizable logics satisfying a bound condition is algebraizable.

Theorem 4.4. (Finiteness Preservation) Let $\mathcal{F} = \{\mathcal{L}_i\}_{i \in I}$ be a family of algebraizable logics, where $\mathcal{L}_i = \langle \Sigma^i, \vdash_{\mathcal{L}_i} \rangle$ for every $i \in I$, and I is a non-empty set. Assume that $\mathcal{F}$ has the following property: there are natural numbers n and m such that, for every $i \in I$, there is an algebraizator $\langle \Delta_i, \langle \varepsilon_i, \delta_i \rangle \rangle$ for $\mathcal{L}_i$ such that Δ_i has at most n elements, and $\langle \varepsilon_i, \delta_i \rangle$ has at most m elements. Then, there exists the product in **ACR** of $\mathcal{F}$.

Proof. By hypothesis we can take, for any $i \in I$, finite sequences

- $\Delta_i^1(p_1,p_2)\cdots\Delta_i^n(p_1,p_2)$
- $\langle\varepsilon_i^1(p_1),\delta_i^1(p_1)\rangle\cdots\langle\varepsilon_i^m(p_1),\delta_i^m(p_1)\rangle$

such that $\langle\Delta_i,\langle\varepsilon_i,\delta_i\rangle\rangle$ is an algebraizator for $\mathcal{L}_i$, where

- $\Delta_i = \{\Delta_i^1(p_1,p_2),\ldots,\Delta_i^n(p_1,p_2)\}$
- $\langle\varepsilon_i,\delta_i\rangle = \{\langle\varepsilon_i^1(p_1),\delta_i^1(p_1)\rangle,\ldots,\langle\varepsilon_i^m(p_1),\delta_i^m(p_1)\rangle\}$, for every $i \in I$.

In fact, it is enough to consider, for every $i \in I$, an algebraizator with at most n elements in Δ_i and at most m elements in $\langle\varepsilon_i,\delta_i\rangle$ and list their elements, repeating, if necessary, some elements, in order to define sequences of length n and m, respectively. Now, consider the product $\langle\mathcal{L}^{\mathcal{F}},\{\pi_i\}_{i\in I}\rangle$ in **CR** of the family $\mathcal{F}$ (recall the proof of Proposition 3.3), and define the following formulas in $L(\Sigma^{\mathcal{F}})$:

- $\Delta_{\mathcal{F}}^j(p_1,p_2) = (\Delta_i^j(p_1,p_2))_{i\in I}(p_1,p_2)$ for $1 \leq j \leq n$;
- $\varepsilon_{\mathcal{F}}^j(p_1) = (\varepsilon_i^j(p_1))_{i\in I}(p_1)$ for $1 \leq j \leq m$;
- $\delta_{\mathcal{F}}^j(p_1) = (\delta_i^j(p_1))_{i\in I}(p_1)$ for $1 \leq j \leq m$.[1]

Finally, let:

- $\Delta_{\mathcal{F}} = \{\Delta_{\mathcal{F}}^j(p_1,p_2) : 1 \leq j \leq n\}$
- $\langle\varepsilon_{\mathcal{F}},\delta_{\mathcal{F}}\rangle = \{\langle\varepsilon_{\mathcal{F}}^j(p_1),\delta_{\mathcal{F}}^j(p_1)\rangle : 1 \leq j \leq m\}$.

We will show that $\langle\Delta_{\mathcal{F}},\langle\varepsilon_{\mathcal{F}},\delta_{\mathcal{F}}\rangle\rangle$ is an algebraizator for $\mathcal{L}^{\mathcal{F}}$ (cf. Definition 4.1).

(1.) $\vdash_{\mathcal{F}} \varphi\Delta_{\mathcal{F}}\varphi$.
Since $\langle\Delta_i,\langle\varepsilon_i,\delta_i\rangle\rangle$ is an algebraizator for $\mathcal{L}_i$ then $\vdash_{\mathcal{L}_i} \widehat{\pi}_i(\varphi)\Delta_i\widehat{\pi}_i(\varphi)$ for every $i \in I$. By definition of $\Delta_{\mathcal{F}}$ we have that $\vdash_{\mathcal{L}_i} \widehat{\pi}_i(\varphi\Delta_{\mathcal{F}}\varphi)$ for every $i \in I$ and hence $\vdash_{\mathcal{F}} \varphi\Delta_{\mathcal{F}}\varphi$.

(2.) $\varphi\Delta_{\mathcal{F}}\psi \vdash_{\mathcal{F}} \psi\Delta_{\mathcal{F}}\varphi$.
Since, for every $i \in I$, $\langle\Delta_i,\langle\varepsilon_i,\delta_i\rangle\rangle$ is an algebraizator for $\mathcal{L}_i$ then we know that $\widehat{\pi}_i(\varphi)\Delta_i\widehat{\pi}_i(\psi) \vdash_{\mathcal{L}_i} \widehat{\pi}_i(\psi)\Delta_i\widehat{\pi}_i(\varphi)$. By definition of $\Delta_{\mathcal{F}}$ we have that $\widehat{\pi}_i(\varphi\Delta_{\mathcal{F}}\psi) \vdash_{\mathcal{L}_i} \widehat{\pi}_i(\psi\Delta_{\mathcal{F}}\varphi)$, for every $i \in I$, and hence $\varphi\Delta_{\mathcal{F}}\psi \vdash_{\mathcal{F}} \psi\Delta_{\mathcal{F}}\varphi$.

(3.) $\varphi\Delta_{\mathcal{F}}\psi, \psi\Delta_{\mathcal{F}}\gamma \vdash_{\mathcal{F}} \varphi\Delta_{\mathcal{F}}\gamma$.
Similar to case (ii).

[1] Observe that the family of formulas $(\Delta_i^j(p_1,p_2))_{i\in I}$ is a binary connective of the signature $\Sigma^{\mathcal{F}}$. Analogously, the families $(\varepsilon_i^j(p_1))_{i\in I}$ and $(\delta_i^j(p_1))_{i\in I}$ are unary connectives of $\Sigma^{\mathcal{F}}$. This justifies the apparently redundant notation employed here.

(4.) Since $\mathcal{L}^{\mathcal{F}}$ is structural, it is enough to prove the following:
$p_1\Delta_{\mathcal{F}}p_{k+1},\ldots,p_k\Delta_{\mathcal{F}}p_{2k} \vdash_{\mathcal{F}} c(p_1,\ldots,p_k)\Delta_{\mathcal{F}}c(p_{k+1},\ldots,p_{2k})$ for every $c \in \Sigma_k^{\mathcal{F}}$.
Let $c = (\varphi_i)_{i\in I} \in \Sigma_k^{\mathcal{F}}$.
If $k = 0$, the result follows from clause (i) of Definition 4.1.
Suppose that $k > 0$. By induction on the length of $\varphi = \varphi(p_1,\ldots,p_k)$, it is easy to show that, if $\langle \Delta', \langle \varepsilon', \delta'\rangle\rangle$ is an algebraizator for a logic $\mathcal{L}'$, then

$$\varphi_1\Delta'\psi_1,\ldots,\varphi_k\Delta'\psi_k \vdash_{\mathcal{L}'} \varphi(\varphi_1,\ldots,\varphi_k)\Delta'\varphi(\psi_1,\ldots,\psi_k)$$

for every $\varphi_1,\ldots,\varphi_k$, $\psi_1,\ldots,\psi_k$ in $L(\Sigma')$. In particular, for every $i \in I$, it holds that $p_1\Delta_i p_{k+1},\ldots,p_k\Delta_i p_{2k} \vdash_{\mathcal{L}_i} \varphi_i(p_1,\ldots,p_k)\Delta_i\varphi_i(p_{k+1},\ldots,p_{2k})$, that is,

$$\widehat{\pi}_i(p_1\Delta_{\mathcal{F}}p_{k+1}),\ldots,\widehat{\pi}_i(p_k\Delta_{\mathcal{F}}p_{2k}) \vdash_{\mathcal{L}_i} \widehat{\pi}_i(c(p_1,\ldots,p_k)\Delta_{\mathcal{F}}c(p_{k+1},\ldots,p_{2k})).$$

Then $p_1\Delta_{\mathcal{F}}p_{k+1},\ldots,p_k\Delta_{\mathcal{F}}p_{2k} \vdash_{\mathcal{F}} c(p_1,\ldots,p_k)\Delta_{\mathcal{F}}c(p_{k+1},\ldots,p_{2k})$.

(5.) $\varphi \dashv\vdash_{\mathcal{F}} \varepsilon_{\mathcal{F}}(\varphi)\Delta_{\mathcal{F}}\delta_{\mathcal{F}}(\varphi)$.[2]
Since, for every $i \in I$, $\langle \Delta_i, \langle \varepsilon_i, \delta_i\rangle\rangle$ is an algebraizator for $\mathcal{L}_i$ then $\widehat{\pi}_i(\varphi) \dashv\vdash_{\mathcal{L}_i} \varepsilon_i(\widehat{\pi}_i(\varphi))\Delta_i\delta_i(\widehat{\pi}_i(\varphi))$. By definition of $\langle \Delta_{\mathcal{F}}, \langle \varepsilon_{\mathcal{F}}, \delta_{\mathcal{F}}\rangle\rangle$, we have that $\widehat{\pi}_i(\varphi) \dashv\vdash_{\mathcal{L}_i} \widehat{\pi}_i(\varepsilon_{\mathcal{F}}(\varphi)\Delta_{\mathcal{F}}\delta_{\mathcal{F}}(\varphi))$, for every $i \in I$. From this, we conclude that

$$\varphi \dashv\vdash_{\mathcal{F}} \varepsilon_{\mathcal{F}}(\varphi)\Delta_{\mathcal{F}}\delta_{\mathcal{F}}(\varphi).$$

This shows that $\langle \Delta, \langle \varepsilon, \delta\rangle\rangle$ is an algebraizator for $\mathcal{L}^{\mathcal{F}}$.

Finally, we must prove that $\langle \mathcal{L}^{\mathcal{F}}, \{\pi_i\}_{i\in I}\rangle$ is the product in **ACR** of the family $\mathcal{F}$. Using Remark 4.3, it is clear that every projection π_i is a **ACR**-morphism. Suppose that $\mathcal{L}'$ is an algebraizable logic having a **ACR**-morphism $\mathcal{L}'\xrightarrow{f_i}\mathcal{L}_i$, for every $i \in I$, and let $\langle \Delta', \langle \varepsilon', \delta'\rangle\rangle$ be an algebraizator for $\mathcal{L}'$. Using the universal property of $\langle \mathcal{L}^{\mathcal{F}}, \{\pi_i\}_{i\in I}\rangle$ in **CR**, we obtain a **CR**-morphism $\mathcal{L}'\xrightarrow{f}\mathcal{L}^{\mathcal{F}}$ such that, in **CR**, $f_i = \pi_i \centerdot f$ for every $i \in I$. Since $p_1\widehat{f}_i(\Delta')p_2 \vdash_{\mathcal{L}_i} p_1\Delta_i p_2$ for every $i \in I$, then $\widehat{\pi}_i(p_1\widehat{f}(\Delta')p_2) \vdash_{\mathcal{F}} \widehat{\pi}_i(p_1\Delta_{\mathcal{F}}p_2)$ for every $i \in I$, by Lemma 2.10, thus $p_1\widehat{f}(\Delta')p_2 \vdash_{\mathcal{F}} p_1\Delta_{\mathcal{F}}p_2$. Analogously we prove that $p_1\Delta_{\mathcal{F}}p_2 \vdash_{\mathcal{F}} p_1\widehat{f}(\Delta')p_2$. Using Remark 4.3, this shows that f is a **ACR**-morphism such that, in **ACR**, $f_i = \pi_i \centerdot f$ for every $i \in I$. The uniqueness of f follows from the universal property of $\langle \mathcal{L}^{\mathcal{F}}, \{\pi_i\}_{i\in I}\rangle$ in **CR**. ■

5 POSSIBLE-TRANSLATIONS SEMANTICS, AND WHEN IT CAN BE REPLACED BY JUST ONE TRANSLATION

Combining and factoring logics are two sides of the same coin. Besides the interest on the synthesis of given logics by means of a combination procedure in order to obtain a new logic (as is the case, for instance, when fibring logics), it is also interesting to split a logic into a family of other (hopefully simpler) logics. This kind

[2]Here $\Delta \dashv\vdash_{\mathcal{L}} \Gamma$ denotes that $\Delta \vdash_{\mathcal{L}} \Gamma$ and $\Gamma \vdash_{\mathcal{L}} \Delta$.

of 'reverse' technique is what is called *splitting logics*, as opposite to the process of *splicing logics* (cf. [10]; see also [11]). In this section we provide a categorial characterization of the process of splitting logics called *possible-translations semantics* (cf. [9]). Possible-translations semantics is a tool for assigning semantical interpretations to logics in general, but this interpretation is done in such a way that a factoring or splitting of the logic in terms of simpler logics is obtained.

We begin by adapting the original definitions of [9] to our formalism.

Definition 5.1. Let $\mathcal{L} = \langle \Sigma, \vdash_{\mathcal{L}} \rangle$ be a logic, and let $\{\mathcal{L}_i\}_{i \in I}$ be a family of logics such that I is a non-empty set and $\mathcal{L}_i = \langle \Sigma^i, \vdash_{\mathcal{L}_i} \rangle$ for every $i \in I$. Let $\mathcal{L} \xrightarrow{f_i} \mathcal{L}^i$ be a **CR**-morphism for every $i \in I$. Then $P = \langle \{\mathcal{L}_i\}_{i \in I}, \{f_i\}_{i \in I} \rangle$ is a *possible-translations semantics for* $\mathcal{L}$ (in short, a **PTS**) if, for every $\Gamma \cup \{\varphi\} \subseteq L(\Sigma)$,

$$\Gamma \vdash_{\mathcal{L}} \varphi \;\text{ iff there is a finite set } \Delta \subseteq \Gamma \text{ such that } \widehat{f_i}(\Delta) \vdash_{\mathcal{L}_i} \widehat{f_i}(\varphi) \;\text{ for every } i \in I.$$

The meaning of having a **PTS** for a logic $\mathcal{L}$ is that $\mathcal{L}$ splits into the family $\{\mathcal{L}_i\}_{i \in I}$ through the translations $\{f_i\}_{i \in I}$.

Inspired by [14] we say that a **CR**-morphism $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ is a *conservative translation* if, for every $\Gamma \cup \{\varphi\} \subseteq L(\Sigma)$,

$$\Gamma \vdash_{\mathcal{L}} \varphi \;\text{ iff }\; \widehat{f}(\Gamma) \vdash_{\mathcal{L}'} \widehat{f}(\varphi).$$

Each logic $\mathcal{L}_i$ is called a *traduct* of $\mathcal{L}$.

Using the results stated in the previous sections, we can characterize **PTS**s in categorial terms.

We show below that possible-translations semantics for a logic induces conservative translations over products of families of logics in **CR**, and vice-versa. Such (induced) conservative translations turn out to be apt to extend the method of finite algebraizability with interesting applications, as shown in Section 6.

Theorem 5.2. Given a possible-translations semantics for a logic $\mathcal{L}$, there exists a conservative translation $\mathcal{L} \xrightarrow{f} \mathcal{L}'$, where $\mathcal{L}'$ is a product in **CR** of some family of logics indexed by a non-empty set. Conversely, every conservative translation $\mathcal{L} \xrightarrow{f} \mathcal{L}'$, where $\mathcal{L}'$ is a product in **CR** of some family of logics indexed by a non-empty set, induces a possible-translations semantics for $\mathcal{L}$.

Proof. Let $P = \langle \{\mathcal{L}_i\}_{i \in I}, \{f_i\}_{i \in I} \rangle$ be a possible-translations semantics for $\mathcal{L}$, and consider the product $\langle \mathcal{L}^{\mathcal{F}}, \{\pi_i\}_{i \in I} \rangle$ of the family $\mathcal{F} = \{\mathcal{L}_i\}_{i \in I}$ in the category **CR** (cf. Proposition 3.3). By the universal property of products, there exists a unique **CR**-morphism $\mathcal{L} \xrightarrow{\mathbf{t}(P)} \mathcal{L}^{\mathcal{F}}$ such that $f_i = \pi_i . \mathbf{t}(P)$ for every $i \in I$. We claim that $\mathbf{t}(P)$ is a conservative translation that encodes P. Since $\mathbf{t}(P)$ is a translation (because it is a **CR**-morphism), it is enough to show that, for every $\Gamma \cup \{\varphi\} \subseteq L(\Sigma)$,

$$\widehat{\mathbf{t}(P)}(\Gamma) \vdash_{\mathcal{F}} \widehat{\mathbf{t}(P)}(\varphi) \;\text{ implies that }\; \Gamma \vdash_{\mathcal{L}} \varphi.$$

Assume that $\widehat{\mathbf{t}(P)}(\Gamma) \vdash_{\mathcal{F}} \widehat{\mathbf{t}(P)}(\varphi)$. Then, by definition of $\mathcal{L}^{\mathcal{F}}$, there exists a finite set $\Delta \subseteq \Gamma$ such that

$$\widehat{\pi}_i(\widehat{\mathbf{t}(P)}(\Delta)) \vdash_{\mathcal{L}_i} \widehat{\pi}_i(\widehat{\mathbf{t}(P)}(\varphi))$$

for every $i \in I$ and then, using Lemma 2.10 and the fact that $f_i = \pi_i \centerdot \mathbf{t}(P)$ for every $i \in I$, we have that $\widehat{f}_i(\Delta) \vdash_{\mathcal{L}_i} \widehat{f}_i(\varphi)$ for every $i \in I$. Since P is a possible-translations semantics for $\mathcal{L}$, we obtain that $\Gamma \vdash_{\mathcal{L}} \varphi$ and then $\mathbf{t}(P)$ is a conservative translation. Clearly, $\mathbf{t}(P)$, together with its codomain $\mathbf{L}(P) = \mathcal{L}^{\mathcal{F}}$ codifies P: every logic $\mathcal{L}_i$ is obtained as the codomain of π_i, and every translation f_i is obtained as $f_i = \pi_i \centerdot \mathbf{t}(P)$.

Conversely, let $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ be a conservative translation, where $\mathcal{L}'$ is a product in **CR** of a family $\{\mathcal{L}_i\}_{i \in I}$ of logics with projections π_i for every $i \in I$, such that I is a non-empty set. Let $f_i = \pi_i \centerdot f$ for every $i \in I$, and define $\mathbf{PT}(f) = \langle \{\mathcal{L}_i\}_{i \in I}, \{f_i\}_{i \in I} \rangle$. We will show that $\mathbf{PT}(f)$ is a **PTS** for $\mathcal{L}$ such that $\mathbf{t}(\mathbf{PT}(f)) = f$. Thus, let $\Gamma \cup \{\varphi\} \subseteq L(\Sigma)$. Since $\mathcal{L}$ satisfies Finitariness and every f_i is a translation, $\Gamma \vdash_{\mathcal{L}} \varphi$ implies that there is a finite set $\Delta \subseteq \Gamma$ such that $\widehat{f}_i(\Delta) \vdash_{\mathcal{L}_i} \widehat{f}_i(\varphi)$ for every $i \in I$. On the other hand, suppose that there exists a finite set $\Delta \subseteq \Gamma$ such that $\widehat{f}_i(\Delta) \vdash_{\mathcal{L}_i} \widehat{f}_i(\varphi)$ for every $i \in I$. By Lemma 2.10,

$$\widehat{\pi}_i(\widehat{f}(\Delta)) \vdash_{\mathcal{L}_i} \widehat{\pi}_i(\widehat{f}(\varphi))$$

for every $i \in I$, where $\widehat{f}(\Delta) \subseteq \widehat{f}(\Gamma)$ is finite. Then $\widehat{f}(\Gamma) \vdash_{\mathcal{F}} \widehat{f}(\varphi)$ and so $\Gamma \vdash_{\mathcal{L}} \varphi$, by the fact that f is conservative. This shows that $\mathbf{PT}(f)$ is a **PTS** for $\mathcal{L}$. Clearly, we recover the information about f and $\mathcal{L}'$ from $\mathbf{PT}(f)$: in fact, $f = \mathbf{t}(\mathbf{PT}(f))$ and $\mathcal{L}'$ is the product of a family of logics of $\mathbf{PT}(f)$. Finally, it is clear that, if P is a **PTS** for $\mathcal{L}$ then $\mathbf{PT}(\mathbf{t}(P)) = P$. This concludes the proof. ∎

6 ALGEBRAIZING LOGICS VIA POSSIBLE-TRANSLATIONS SEMANTICS

The last result (Theorem 5.2) give support to a method for algebraizing logics using a **PTS**s which extends the well-known method of finite algebraizability of [2].

Definition 6.1. A *possible-translations algebraic semantics* (in short, a **PTAS**) for a propositional logic $\mathcal{L}$ is a triple $\mathbf{PA} = \langle \{\mathcal{L}_i\}_{i \in I}, \{\mathcal{A}_i\}_{i \in I}, \{f_i\}_{i \in I} \rangle$ such that:
(i) $P = \langle \{\mathcal{L}_i\}_{i \in I}, \{f_i\}_{i \in I} \rangle$ is a possible-translations semantics for $\mathcal{L}$;
(ii) For each $i \in I$, $\mathcal{A}_i = \langle \Delta_i, \langle \varepsilon_i, \delta_i \rangle \rangle$ is an algebraizator for $\mathcal{L}_i$ (cf. Definition 4.1).

The idea, proposed and studied in [5] and [4], is the following: consider a propositional logic $\mathcal{L}$, and let $P = \langle \{\mathcal{L}_i\}_{i \in I}, \{f_i\}_{i \in I} \rangle$ be a **PTS** for $\mathcal{L}$. Suppose that every $\mathcal{L}_i$ is algebraizable, and assume that the family $\mathcal{F} = \{\mathcal{L}_i\}_{i \in I}$ satisfies the bound condition of Theorem 4.4. Then, by Theorem 5.2, the product $\langle \mathcal{L}^{\mathcal{F}}, \{\pi_i\}_{i \in I} \rangle$ of the family $\mathcal{F}$ in **ACR** encodes P. Moreover, it is possible to build an algebraizator for $\mathcal{L}^{\mathcal{F}}$ from a bounded (in the sense of Theorem 4.4) family of algebraizators for $\mathcal{F}$. This shows that there exists a conservative translation $\mathcal{L} \xrightarrow{f} \mathcal{L}^{\mathcal{F}}$, where $\mathcal{L}^{\mathcal{F}}$ is an algebraizable logic. The conservative translation f is a link between $\mathcal{L}$ and $\mathcal{L}^{\mathcal{F}}$

that preserves derivability and so, using the algebraization for $\mathcal{L}^{\mathcal{F}}$, one obtains a kind of 'remote' algebraization for $\mathcal{L}$: in order to algebraically analyze $\mathcal{L}$, it is sufficient to conservatively translate $\mathcal{L}$ into $\mathcal{L}^{\mathcal{F}}$ and then analyze the result using the algebraic resources of $\mathcal{L}^{\mathcal{F}}$.

A natural generalization of **PTAS**s is the following:

Definition 6.2. Let $\mathcal{L} = \langle \Sigma, \vdash_{\mathcal{L}} \rangle$ be a propositional logic and let

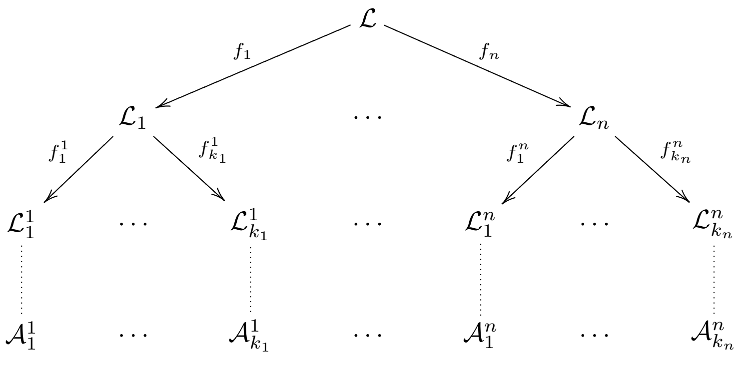
$$P = \langle \{\mathcal{L}_i\}_{i \in I}, \{f_i\}_{i \in I} \rangle$$

be a **PTS** for $\mathcal{L}$. We define recursively the following:
(i) P is a **PTAS** of level 0 if every traduct $\mathcal{L}_i$ admits an algebraizator $\mathcal{A}_i$;
(ii) P is a **PTAS** of level $n+1$ if every $\mathcal{L}_i$ admits a **PTAS** of level n, for some $n \in \omega$.

If $P = \langle \{\mathcal{L}_i\}_{i \in I}, \{f_i\}_{i \in I} \rangle$ is a **PTAS** for $\mathcal{L}$ of level n, then each $\mathcal{L}_i$ is a traduct of level 0; the traducts of each $\mathcal{L}_i$ are traducts of level 1, and so on. Therefore, just the traducts of level n are algebraizable in the sense of Blok-Pigozzi. Thus, the algebraizators of P are the algebraizators of the traducts of level n. Note that a **PTAS** of level 0 for $\mathcal{L}$ is equivalent to a **PTAS** for $\mathcal{L}$, in the sense of Definition 6.1. Thus, Definition 6.2 generalizes Definition 6.1. However, the next result shows that this generalization is innocuous, under certain hypotheses.

Theorem 6.3. Let $P = \langle \{\mathcal{L}_i\}_{i \in I}, \{f_i\}_{i \in I} \rangle$ a **PTAS** of level $n > 1$ for a propositional logic $\mathcal{L}$ such that the algebraizators (of the traducts of level n) are globally bounded in the sense of Theorem 4.4. Then P can be transformed in a **PTAS** for $\mathcal{L}$ of level 0 and bounded algebraizators.

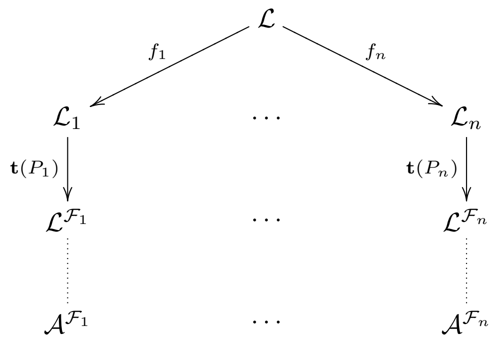
Proof. Clearly, it is enough to show how to obtain a **PTAS** of level 0 from a given **PTAS** of level 1. Thus, let $P = \langle \{\mathcal{L}_i\}_{i \in I}, \{f_i\}_{i \in I} \rangle$ be a **PTAS** of level 1 for a propositional logic $\mathcal{L}$. Then P is of the form displayed below (for simplicity, in the figure below we consider finite sets of indices everywhere).



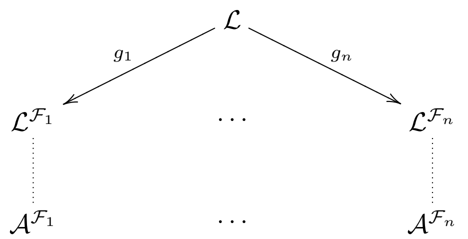
In the general case, for each $i \in I$ the logic $\mathcal{L}_i$ admits a **PTAS**, say

$$\mathbf{PA}_i = \langle \{\mathcal{L}_j^i\}_{j \in J^i}, \{\mathcal{A}_j^i\}_{j \in J^i}, \{f_j^i\}_{j \in J^i} \rangle$$

such that the cardinals of the algebraizators involved are globally bounded, in the sense of Theorem 4.4; that is, there are bounds r and m for the cardinal of every algebraizator $\mathcal{A}^i_j$. For each $i \in I$ let $P_i = \langle \{\mathcal{L}^i_j\}_{j \in J^i}, \{f^i_j\}_{j \in J^i} \rangle$ be the **PTS** for $\mathcal{L}_i$ obtained from $\mathbf{PA}_i$, and let $\mathcal{F}_i = \{\mathcal{L}^i_j\}_{j \in J^i}$. By the proof of Theorem 4.4, if $\langle \mathcal{L}^{\mathcal{F}_i}, \{\pi^i_j\}_{j \in J^i} \rangle$ is the product in the category **CR** of the family $\mathcal{F}_i$ then it is, in fact, the product in the category **ACR** of the family $\mathcal{F}_i$, and thus $\mathcal{L}^{\mathcal{F}_i}$ admits an algebraizator $\mathcal{A}^{\mathcal{F}_i} = \langle \Delta_{\mathcal{F}_i}, \langle \varepsilon_{\mathcal{F}_i}, \delta_{\mathcal{F}_i} \rangle \rangle$ bounded by r and m. For each $i \in I$ let $\mathcal{L}_i \overset{\mathbf{t}(P_i)}{\longrightarrow} \mathcal{L}^{\mathcal{F}_i}$ be the conservative translation such that $f^i_j = \pi^i_j \centerdot \mathbf{t}(P_i)$ for every $j \in J^i$, guaranteed by the proof of Theorem 5.2. In the finite case, we obtain the following figure:



Now, for every $i \in I$ let $g_i = \mathbf{t}(P_i) \centerdot f_i$. Then $\mathcal{L} \overset{g_i}{\longrightarrow} \mathcal{L}^{\mathcal{F}_i}$ is a morphism in **CR**, for every $i \in I$. In the finite case, the new situation is displayed below.



Thus, $\bar{P} = \langle \{\mathcal{L}^{\mathcal{F}_i}\}_{i \in I}, \{g_i\}_{i \in I} \rangle$ is a **PTAS** of level 0 for $\mathcal{L}$, because $\bar{P}$ is a **PTS** for $\mathcal{L}$ such that every traduct $\mathcal{L}^{\mathcal{F}_i}$ is algebraizable. Moreover, the algebraizators of $\bar{P}$ are bounded by r and m in the sense of Theorem 4.4. ■

The next result is an application of Theorem 6.3 that shows that the logic C_{Lim} (introduced in [6]) has a possible-translations algebraic semantics. In that article, the authors show that C_{Lim} is characterized by a possible-translations semantics whose traducts are the paraconsistent logics C_n. Since these logics are characterized by a possible-translations semantics where the traducts are all coincident with the logic **LFI1** (see [5]) , the result will be obtained by compounding these facts.

Theorem 6.4. The paraconsistent logic C_{Lim} has a **PTAS**.

Proof. Theorem 6.3 together with the facts above. ■

The last result is a concrete application of Theorem 6.3. Possible-translations semantics is used (cf. [9] and [18]) to obtain a new semantics for the paraconsistent systems C_n introduced in [13]. Such semantics permits characterizing the logics C_n in terms of a family $\mathcal{F}_n$ of copies of the three-valued logic **LFI1** (cf. [7]). The latter is equivalent to the three-valued paraconsistent logic J_3 (introduced in [15]).[3] It turns out that J_3, **LFI1** and the three-valued Łukasiewicz logic L_3 are all finitely algebraizable with the same equivalent quasivariety, to wit, the quasivariety of the three-valued Moisil algebras, as shown in [3], p. 43.

It is clear then that every traduct J_3 (or **LFI1**) in the family $\mathcal{F}_n$ is finitely algebraizable and so the algebraizators of the family $\mathcal{F}_n$ obviously satisfy the bound condition of Theorem 4.4. As a consequence of this, the product $\mathcal{L}^{\mathcal{F}_n}$ of the family $\mathcal{F}_n$ is also algebraizable, as argued in [5].

Furthermore, as a consequence of Theorem 5.2, there exists a conservative translation from each C_n into $\mathcal{L}^{\mathcal{F}_n}$.

This shows that our categorial characterization extends the concept of finite algebraizability in very adequate terms, offering a non *ad hoc* solution to the question of algebraizing logics in general, as it amply extends the method of [2]. Other interesting questions, as the characterization of logics having the Craig interpolation property for their consequence relations, can be recast as a challenging problem in our setting: indeed, it is known (see [17] page 43 for a discussion) that a logic enjoying the deduction-detachment theorem has the Craig interpolation property iff its algebraization has the amalgamation property. Since the amalgamation property can be seen as a universal construction in our (sub)category **CR** of algebraizable logics, it remains as an open problem to know whether this would correspond to any form of Craig interpolation property.

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[3] It is noticeable that J_3 and **LFI1** are also equivalent in terms of consequence relations with the system **CLuNs** (cf. [1]) and with the system Φ_v, introduced in [19].

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