Why are combined modal logics so robustly undecidable?

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One of the main reasons for the success of modal logics in computer science is their unusual robust decidability. Indeed, standard modal logics like polymodal $\mathbf{K}$, $\mathbf{S4}$, and $\mathbf{S5}$, temporal logics like $\mathbf{LTL}$ and $\mathbf{CTL}$, dynamic logics like $\mathbf{PDL}$, epistemic logics like $\mathbf{S5}$ with common knowledge, description logics like $\mathbf{ALC}$ and $\mathbf{SHI}Q$, and modal spatial logics like $\mathbf{S4_{mu}}$, are decidable in EXPTIME [7, 4].

However, modern applications of modal languages often require rather complex formal models and corresponding languages that are capable of reflecting different features of an application domain. It is not sufficient to work with just one sort of modal operators (say, epistemic or temporal) but subtle combinations of different families of modal operators are required. For example,

- to analyze the behaviour of a multi-agent distributed system we may need a formalism containing both epistemic operators for capturing knowledge of agents and temporal operators for taking care of the development of this knowledge in time. In other words, we should construct a suitable combination of epistemic and temporal logics [2].

- To describe the behaviour of spatial objects (regions) changing over time we may need combinations of spatial modal logics with temporal logics;

- to analyze the dynamics of ontologies and the knowledge of agents about ontologies we may need combinations of description logics (as the underlying language for ontologies) with dynamic modal logics and epistemic logics.

- Fragments of first-order logic may be required to describe an application domain. In this case, instead of propositional modal logics, first-order modal logics are required; i.e., we have to combine (fragments) of first-order logic with propositional modal logics.

While decidability is preserved under forming combinations of modal logics without interaction axioms or constraints (i.e., fusions) [8, 1] this situation changes drastically as soon as some kind of interaction between the modalities is required. In contrast to standard modal logics, combined modal logics often exhibit rather nasty computational properties, and the standard toolkit of modal logic, e.g.,

- proving the finite model property by some kind of filtration;

- proving a variant of the tree-model property and applying automata-based decision procedures [7, 4];

- providing an embedding into a decidable fragment of first-order logic [4]
is no longer directly available. In fact, straightforward constructions of combined modal logics from
simple one-dimensional ones will almost certainly result in computationally useless ‘monsters’.

The aim of this presentation is twofold: (i) to discuss possible explanations for this phenomenon -
the robust undecidability of combined modal logics - and (ii) to use the insights gained from this dis-
cussion in order to provide methodologies for constructing computationally well-behaved combined
modal logics.

We will argue that the high computational complexity or even undecidability of most combined
modal logics is explained by the two- or many-dimensional structure of their intended models [3]:

• The fact that almost all three-dimensional modal logics are undecidable can be intuitively ex-
plained by the undecidability of the product $S5^3$ and its relation to the undecidable 3-variable
fragment of first-order logic.

• Two-dimensional modal logics are usually at least NEXPTIME-hard. This can by intuitively
explained by the NEXPTIME-completeness of the product $S5^2$ and its relation to the 2-variable
fragment of first-order logic. However, the computational properties of two-dimensional modal
logics depend - in a way not yet completely understood - on the geometry of the models of the
components. We will present partial results for two-dimensional modal logics as well as open
problems based on [3].

• Products of $S5$ with CTL* and CTL may be regarded as $2 \frac{1}{2}$-dimensional modal logics and show
again the computational significance of the move from two to three dimensions: the product
with CTL* is undecidable while the product with CTL (or LTL) is decidable [5].

• The decidability/undecidability of fragments of first-order modal logics is closely related to the
number of free variables allowed withing formulas starting with a modal operator; i.e., formulas
of the form $\Box \varphi(x)$. While monodic fragments - which allows for just one free such variable -
often exhibits ‘good’ computational properties, two variables usually lead to undecidable frag-
ments already [3]. We will give an overview of explanations for this and discuss open problems.

• Finally, we discuss the method of E-connections to produce decidable and expressive combina-
tions of modal logics of many dimensions [6].

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